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TWO-DIMENSIONAL NUMERICAL MODEL FOR TIDAL MOTION IN THE SEA

1. Basic hydrodynamic equations for tidal problems

This study is based upon the general equation of continuity for an incompressible fluid

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{1}$$

and the general equations of motion for a viscous fluid

$$\begin{aligned}
&\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = X - \frac{1}{\varrho}\frac{\partial P}{\partial x} + A\Delta U\\
&\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = Y - \frac{1}{\varrho}\frac{\partial P}{\partial y} + A\Delta V\\
&\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = Z - \frac{1}{\varrho}\frac{\partial P}{\partial z} + A\Delta W
\end{aligned} \tag{2}$$

Integrating vertically equation (1) from — D (x, y) to ζ (x, y) with the following boundary conditions

$$W(\zeta) = U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial t}$$

$$U(x, y, -D) = V(x, y, -D) = W(x, y, -D) = 0$$

and introducing mean values

$$\begin{aligned}
u &= \frac{1}{D+\zeta}\int_{-D}^{\zeta} U\,dz\\
v &= \frac{1}{D+\zeta}\int_{-D}^{\zeta} V\,dz
\end{aligned} \tag{3}$$

we obtain equation of continuity for tidal problems in the form of

$$\frac{\partial \zeta}{\partial t} + \frac{\partial\, [(D+\zeta)\, u]}{\partial x} + \frac{\partial\, [(D+\zeta)\, w]}{\partial y} = 0 \tag{4}$$

Considering tidal motion in an adjacent sea, actual forces in system of equations (2) are Coriolis force and gravity g. The components of the first force in the x, y directions are fV, — fU, respectively; the component in z direction being very small is neglected. Convective terms can also be neglected [2, 4] equations (2) are reduced to

$$\begin{aligned} &\frac{\partial U}{\partial t} - fV + \frac{1}{\varrho}\frac{\partial P}{\partial x} - A\Delta U = 0 \\ &\frac{\partial V}{\partial t} + fU + \frac{1}{\varrho}\frac{\partial P}{\partial y} - A\Delta V = 0 \end{aligned} \tag{5}$$

$$g + \frac{1}{\varrho}\frac{\partial P}{\partial z} = 0 \tag{6}$$

Integrating (6) vertically from an arbitrary point z in the water to the point ζ (x, y) on the free surface, we derive

$$P(z) = Pa + \varrho\, g\, (\zeta - z) \tag{7}$$

Neglecting further the atmospheric pressure, we can write the derivatives of the pressure in horizontal plane as

$$\begin{aligned} &\frac{\partial P}{\partial x} = \varrho\, g\, \frac{\partial \zeta}{\partial x} \\ &\frac{\partial P}{\partial y} = \varrho\, g\, \frac{\partial \zeta}{\partial y} \end{aligned} \tag{8}$$

Introducing the relations (8) into equations (5) then integrating vertically from — D (x, y) to ζ (x, y) with the use of (3) we obtain

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} - f\, v - A\Delta u = 0 \tag{9}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial y} + f\, u - A\Delta v = 0 \tag{10}$$

As the bottom stress is proportional to the velocity, it affects the behaviour of tidal motion considerably. Some results on this effect are given hereafter.

The components of the force in x, y directions are [1]

$$F_{bx} = -r\frac{\sqrt{u^2+v^2}}{D+\zeta}\, u$$

$$F_{by} = -r\frac{\sqrt{u^2+v^2}}{D+\zeta}\, v$$

Introduction of these components into equations (9) — (10) results in

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} - fv - \frac{r\sqrt{u^2+v^2}}{D+\zeta}u - A\,\Delta\,u = 0 \tag{11}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial y} + fu + \frac{r\sqrt{u^2+v^2}}{D+\zeta}v - A\,\Delta\,v = 0 \tag{12}$$

Equation of motion (11) — (12) together with the equation of continuity

$$\frac{\partial \zeta}{\partial t} + \frac{\partial\,[(D+\zeta)\,u]}{\partial x} + \frac{\partial\,[(D+\zeta)\,v]}{\partial y} = 0 \tag{13}$$

are the basic equations for mathematical treatment of tidal problems.

Assuming that the tidal motion of the sea is generated from an initial state of rest, a solution of equations (11) — (13) is sought in which

$$\zeta = u = v = 0 \text{ at } - t = 0$$

everywhere. In addition to the above, appropriate conditions have to be satisfied along the lateral boundaries of the sea for all time. Thus, along coastal boundaries, zero normal horizontal flow requires

$$u \cos\psi + v \sin\psi = 0$$

where ψ denotes the angle between the normal, directed out of the domain, and the x-axis. On the other hand, at any point M on an open boundary, the ζ level is prescribed as a function of time

$$\zeta|\text{open boundary} = \zeta\,(x, y, t)$$

2. Filtering operator

In the numerical integration of hydrodynamic equations, high frequency components may arise from truncation error. Excess of the components may cause instability or extension of convergent process. One may expect to reduce the instability and to improve the convergence by suppressing the shortest wavelength components. To do this the so-called smoothing operator or numerical filter is introduced. The operator may be constructed so that it eliminates the shortest wavelength components and does not seriously affect the phenomenon under consideration.

In this paper, a two-dimensional filtering operator is introduced (see Fig. 1):

$$\tilde{F}_{m,n} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} F_{m-2,n+2} & F_{m,n+2} & F_{m+2,n+2} \\ F_{m-2,n} & F_{m,n} & F_{m+2,n} \\ F_{m-2,n-2} & F_{m,n-2} & F_{m+2,n-2} \end{bmatrix} \tag{14}$$

It may be proved by Shuman's methods [6] that the filter (14) introduces neither phase shift nor change of wave number. But it eliminates the shortest wavelength components and improves the computational stability.

Fig. 1. Space field of a smoothing point (m, n)

Ryc. 1. Pole wygładzania punktu (m, n)

From (14) it can be deduced

$$\left(\frac{\partial \tilde{F}}{\partial x}\right)_{m,n} = \frac{1}{32\,\Delta S} \sum_{j'=-1}^{1} \sum_{j=0}^{1} \left(F_{m+1+2j,n+2j'} - F_{m-1-2j,n+2j'}\right) \tag{15}$$

$$\left(\frac{\partial \tilde{F}}{\partial y}\right)_{m,n} = \frac{1}{32\,\Delta S} \sum_{j'=-1}^{1} \sum_{j=0}^{1} \left(F_{m+2j',n+1-2j} - F_{m+2j',n-1-2j}\right) \tag{16}$$

$$\tilde{\tilde{F}}_{m,n} = \frac{1}{64} \sum_{j=-2}^{1} \left[\frac{3}{|2j+1|}\left(F_{m-3,n+2j+1} + F_{m+3,n+2j+1}\right) + \frac{9}{|2j+1|}\left(F_{m-1,n+2j+1} + F_{m+1,n+2j+1}\right)\right] \tag{17}$$

A finite difference scheme with the filtering operator mentioned above is constructed and its stability is discussed hereafter.

3. A finite difference scheme with filtering operator

A computational grid of several interlocking subgrids is used (see Fig. 2).

For this grid, the calculation of ζ on one subgrid requires a knowledge of u, v half of time step removed on a different subgrid.

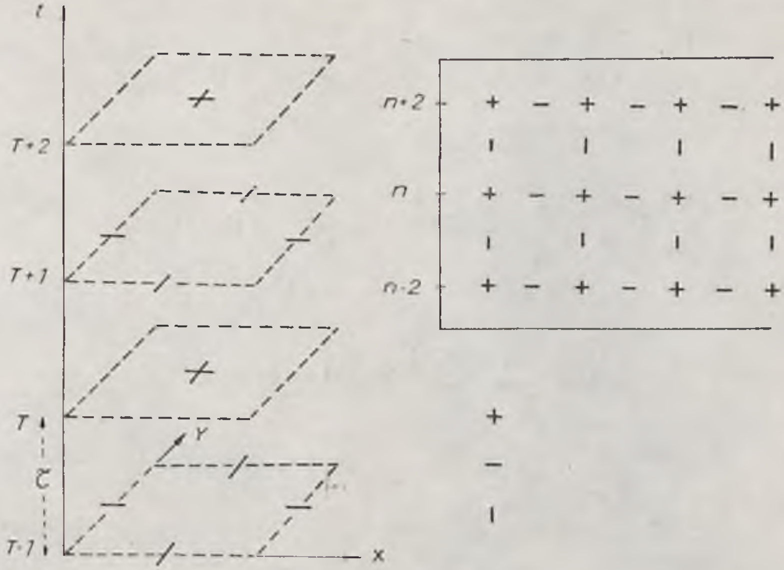

Fig. 2. Computational grid
Ryc. 2. Siatka obliczeń numerycznych

Values of u at v-points and thoses of v at u-points in Coriolis terms are replaced by their mean values as follows

$$\bar{u}_{m,n} = \frac{1}{4}\left(u_{m+1,n+1} + u_{m+1,n-1} + u_{m-1,n-1} + u_{m-1,n+1}\right) \tag{18}$$

$$\bar{v}_{m,n} = \frac{1}{4}\left(v_{m+1,n+1} + v_{m+1,n-1} + v_{m-1,n-1} + v_{m-1,n+1}\right. \tag{19}$$

It is assumed that ζ is very small compared with the depth D. Approximating system of equations (11) — (13) by finite difference equations with the use of relations (14) — (19) we obtain the following numerical model

$$u^{T+1}_{m+1,n} = (1 - 2\,R_x\tau)\,u^{T-1}_{m+1,n} + \frac{f\,\tau}{32}\sum_{j=-2}^{1}\left[\frac{3}{|2\,j+1|}\left(v^{T-1}_{m-2,n+2j+1} + \right.\right.$$

$$\left.\left. + v^{T-1}_{m+4,n+2j+1}\right) + \frac{9}{|2j+1|}\left(v^{T-1}_{m,n+2j+1} + v^{T-1}_{m+2,n+2j+1}\right)\right] -$$

$$- \frac{g\,\tau}{16\,l}\sum_{j'=-1}^{1}\sum_{j=0}^{1}\left(\zeta^{T}_{m+2+2j,n+2j'} - \zeta^{T}_{m-2j,n+2j'} + \frac{A\,\tau}{2\,l^2}\left(u^{T-1}_{m+3,n} + u^{T-1}_{m-1,n} + \right.\right.$$

$$\left.\left. + u^{T-1}_{m+1,n+2} + u^{T-1}_{m+1,n-2} - 4\,u^{T-1}_{m+1,n}\right)\right. \tag{20}$$

$$v^{T+1}_{m,n+1} = (1 - 2 R_y \tau) v^{T-1}_{m,n+1} - \frac{f\tau}{32} \sum_{j=-2}^{1} \Big[\frac{3}{|2j+1|} (u^{T-1}_{m-3,n+2j+2} +$$

$$+ u^{T-1}_{m+3,n+2j+2}) + \frac{9}{|2j+1|} (u^{T-1}_{m-1,n+2j+2} + u^{T-1}_{m+1,n+2j+2}) \Big] -$$

$$- \frac{g\tau}{16 l} \sum_{j'=-1}^{1} \sum_{j=0}^{1} (\zeta^{T}_{m+2j',n+2-2j} - \zeta^{T}_{m+2j',n-2j}) + \frac{A\tau}{2 l^2} (v^{T-1}_{m-2,n+1} + v^{T-1}_{m+2,n+1} +$$

$$+ v^{T-1}_{m,n-1} + v^{T-1}_{m,n+3} - 4 v^{T-1}_{m,n+1}) \quad (21)$$

$$\zeta^{T+2}_{m,n} = \zeta^{T}_{m,n} - \frac{D\tau}{16 l} \sum_{j'=-1}^{1} \sum_{j=0}^{1} \Big[(u^{T+1}_{m+1+2j,n+2j'} - u^{T+1}_{m-1-2j,n+2j'}) +$$

$$+ \sum_{j'=-1}^{1} \sum_{j=0}^{1} (v^{T+1}_{m+2j' n+1-2j} - v^{T+1}_{n+2j',n-1-2j}) \Big] \quad (22)$$

It is seen that, central differences are used for time and space derivatives.

4. Stability

The general solution of the system of equations (20) — (22) can be expressed at the points of the lattice by typical Fourier terms

$$\zeta^{T}_{m,n} = \zeta^* \lambda^T e^{i(m\alpha l + n\beta l)} \quad (23)$$

$$u^{T}_{m,n} = u^* \lambda^T e^{i(m\alpha l + n\beta l)} \quad (24)$$

$$v^{T}_{m,n} = v^* \lambda^T e^{i(m\alpha l + n\beta l)} \quad (25)$$

where $\lambda^T = l^{i\Theta T\tau}$; α, β are wave numbers and ζ^*, u^*, v^* are constants. It is supposed that $R_x = R_y = R =$ const., $D =$ const.

Inserting relations (23) — (25) into equations (20) — (22) the following system, after some rearrangements, is obtained

$$u^{T+1} = \Big[(1 - 2 R \tau) - 2 A (\sin^2 \alpha l + \sin^2 \beta l) \frac{\tau}{l^2} \Big] u^{T-1} +$$

$$+ 2 \gamma^3 f \tau v^{T-1} - 2 i \gamma^2 \frac{g\tau}{l} \sin \alpha l \cdot \zeta^T \quad (26)$$

$$v^{T+1} = \Big[(1 - 2 R \tau) - 2 A (\sin^2 \alpha l + \sin^2 \beta l) \frac{\tau}{l^2} \Big] v^{T-1} -$$

$$- 2\gamma^3 f \tau u^{T-1} - 2 i \gamma^2 \frac{g\tau}{l} \sin \beta l \cdot \zeta^T \quad (27)$$

$$\zeta^{T+2} = 2\,i\,D\,\frac{\tau}{l}\Big[\gamma^2(1 - 2\,R\,\tau)\sin\alpha\,l - 2\gamma^2 f\,\tau\sin\beta\,l -$$

$$- 2\,A\,\gamma^2\sin\alpha\,l\,(\sin^2\alpha\,l + \sin^2\beta\,l)\,\frac{\tau}{l^2}\Big]\,u^{T-1} - 2\,i\,D\,\frac{\tau}{l}\Big[\gamma^2(1 -$$

$$- 2\,R\,\tau)\sin\beta\,l + 2\,\gamma^5 f\,\tau\sin\alpha\,l - 2\,A\gamma^2\sin\beta\,l\,(\sin^2\alpha\,l\sin^2\beta\,l)\,\frac{\tau}{l^2}\Big]\,v^{T-1} +$$

$$+ \Big[1 - 4\,\gamma^4(\sin^2\alpha\,l + \sin^2\beta\,l)\,g\,D\,\frac{\tau}{l^2}\Big]\,\zeta^T \qquad (28)$$

where

$$\gamma = \cos\alpha\,l \cdot \cos\beta\,l$$

System of equations (26) — (28) can be rewritten in vector form as

$$\begin{bmatrix} u^{T+1} \\ v^{T+1} \\ \zeta^{T+2} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_2 & -2\,i\,\gamma^2 g\,\frac{\tau}{l}\sin\alpha\,l \\ -\sigma_2 & \sigma_1 & -2\,i\,\gamma^2 g\,\frac{\tau}{l}\sin\beta\,l \\ -2\,i\,\gamma^2 D\,\frac{\tau}{l}\,\Theta_1 & -2\,i\,\gamma^2 D\,\frac{\tau}{l}\,\Theta_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} u^{T-1} \\ v^{T-1} \\ \zeta^T \end{bmatrix}$$

where

$$\sigma_1 = 1 - 2\,R\,\tau - 2\,A\,(\sin^2\alpha\,l + \sin^2\beta\,l)\,\frac{\tau}{l^2}$$

$$\sigma_2 = 2\,\gamma^3 f\,\tau$$

$$\sigma_3 = 1 - 4\,\gamma^4(\sin^2\alpha\,l + \sin^2\beta\,l)\,g\,D\,\frac{\tau^2}{l^2}$$

$$\Theta_1 = \sigma_1\sin\alpha\,l - \sigma_2\sin\beta\,l$$

$$\Theta_2 = \sigma_1\sin\beta\,l + \sigma_2\sin\alpha\,l$$

The von Neumann stability condition requires that the eigenvalues λ of the amplification matrix G should not exceed unity in absolute value [5]. The eigenvalues can be found from the determinant

$$\begin{vmatrix} \sigma_1 - \lambda^2 & \sigma_2 & -2\,i\,\gamma^2 g\,\frac{\tau}{l}\sin\alpha\,l\,\lambda \\ -\sigma_2 & \sigma_1 - \lambda^2 & -2\,i\,\gamma^2 g\,\frac{\tau}{l}\sin\beta\,l\,\lambda \\ -2\,i\,\gamma^2 D\,\frac{\tau}{l}\,\Theta_1\lambda^{-1} & -2\,i\,\gamma^2 D\,\frac{\tau}{l}\,\Theta_2\lambda^{-1} & \sigma_3 - \lambda^2 \end{vmatrix} = 0 \qquad (29)$$

From (29) the following characteristic equation is deduced

$$(\lambda^2 - 1)\,\{\lambda^4 - 2\,[\sigma_1 - 2\,\gamma^4(\sin^2\alpha\,l + \sin^2\beta\,l)\,gD\,\frac{\tau^2}{l^2}\Big]\,\lambda^2 +$$

$$+ \sigma_1^2 + \sigma_2^2 + \sigma_4\} + \sigma_4 = 0 \qquad (30)$$

where

$$\sigma_4 = 4\,\gamma^4 (\sin^2 \alpha\, l + \sin^2 \beta\, l)\, gD \frac{\tau^2}{l^2} \left[2\, R\, \tau + 2\, A\, (\sin^2 \alpha\, l + \sin^2 \beta\, l) \frac{\tau}{l^2} \right]$$

It is very difficult to find from equation (30) the general conditions for which $|\lambda| \leqslant 1$. But the most important cases can be deduced.

First, the case of $R = f = A = 0$ is discussed. In this case $\sigma_1 = 1$, $\sigma_2 = \sigma_4 = 0$ and equation (30) is reduced to

$$(\lambda^2 - 1) \left\{ \lambda^4 - 2 \left[1 - 2\,\gamma^4 (\sin^2 \alpha\, l + \sin^2 \beta\, l)\, gD \frac{\tau^2}{l^2} \right] \lambda^2 + 1 \right\} = 0 \qquad (31)$$

The roots of this equation are

$$\lambda^2_{1,2} = 1$$

$$\lambda^2_{3,4} = 1 - 2\,\gamma^4 (\sin^2 \alpha\, l + \sin^2 \beta\, l)\, gD \frac{\tau^2}{l^2} \pm$$

$$\pm \sqrt{\left[1 - 2\,\gamma^4 (\sin^2 \alpha\, l + \sin^2 \beta\, l)\, gD \frac{\tau^2}{l^2} \right]^2 - 1}$$

From these, we obtain

$|\lambda_{1,2}| = 1$ and $|\lambda_{3,4}| = 1$ providing that

$$\left[1 - 2\,\gamma^4 (\sin^2 \alpha\, l + \sin^2 \beta\, l)\, gD \frac{\tau^2}{l^2} \right] - 1 \leqslant 0$$

or

$$\tau \leqslant \frac{1}{\sqrt{2gD}} \qquad (32)$$

which is the well-known stability condition of Courant, Friedrichs and Lewy.

In cases where the values of R, f, A are not equal to zero simultaneously, discussion is restricted to the highest frequency component, the interoction of which may cause instability.

The wave of shortest length $L = 4\, l$ for which the grid is capable of discriminating is now considered.

If $L = 4\, l$, $f \neq 0$, $R \neq 0$ and $A \neq$ we have

$$\alpha\, l = \beta\, l = \frac{\pi}{2}, \qquad \sigma_1 = 1 - 2\, R\, \tau - 4\, A \frac{\tau}{l^2}$$

$$\sigma_2 = \sigma_4 = 0$$

Thus, the equation (30) is reduced to

$$(\lambda^2 - 1) \left[\lambda^4 - 2 \left(1 - 2\, R\, \tau - 4\, A \frac{\tau}{l^2} \right) \lambda^2 + \left(1 - 2\, R\, \tau - 4\, A \frac{\tau}{l^2} \right)^2 \right] = 0$$

The roots of this equation are

$$\lambda_{1,2}^2 = 1, \quad \lambda_{3,4}^2 = 1 - 2\left(R + \frac{2A}{l^2}\right)\tau$$

Therefore, we have

$$|\lambda_{1,2}| = 1, \quad |\lambda_{3,4}^2| = \left|1 - 2\left(R + \frac{2A}{l^2}\right)\tau\right| \leqslant 1$$

provided that

$$\tau \leqslant \frac{1}{R + \frac{2A}{l^2}} \tag{33}$$

Special cases can be futher deduced as follows

If $R = 0$, the stability condition (33) is reduced to

$$\tau \leqslant \frac{l^2}{2A} \tag{34}$$

If $A = 0$, it becomes

$$\tau \leqslant \frac{1}{R} \tag{35}$$

If both the values R and A vanish, inequality (33) is always valid, therefore the numerical model (20) — (22) is unconditionally stable for the wave length of 4l.

In this case ($L = 4l$), a weaker stability condition was shown by Kagan [3] however, to obtain this condition a supposition without any discussion was introduced.

5. Computational tests

The stability condition discussed above can be used as a guide line in constructing a stable numerical model. But the practical stability of any model must be verified by test computation.

In this section, computations for a semidiurnal tide are tested in a rectangular basin with constant depth. The results obtained indicate computational stability and rapid convergence.

Figures 3 and 4 give a comparison between rate of convergence for the above difference scheme and that for the identical scheme except that smoothing operator is not involved.

In the figures, rate of convergence is estimated by relative error

$$\varepsilon\,\% = 100 \left|\frac{\zeta^k - \zeta^{k-1}}{\zeta_{max}}\right|$$

where

ζ^k, ζ^{k-1} are $K^{th}$and $(K\text{-}1)^{th}$ iterative approximations to water level functions.

ζ_{max} is maximal elevations of water level.

Clearly, the rate of convergence is improved considerably by the introduction of a smoothing operator.

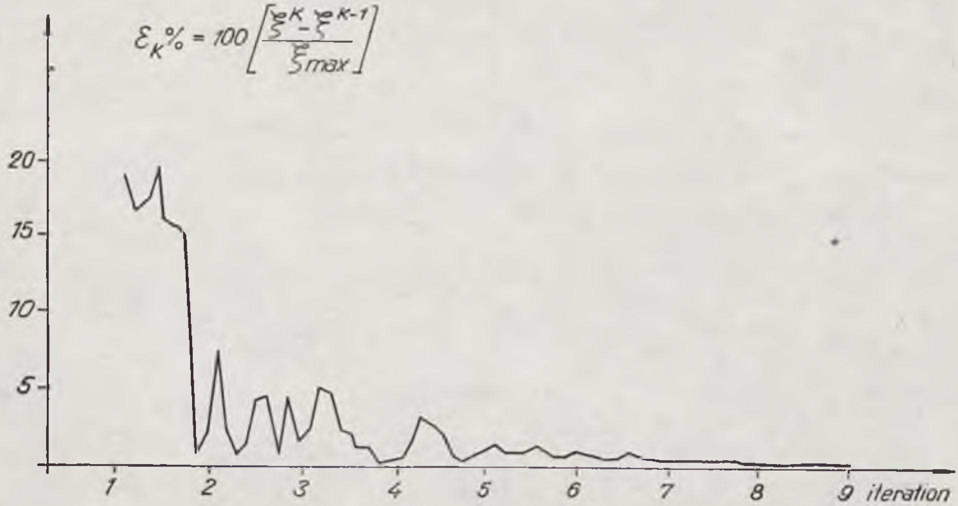

Fig. 3. Convergence relative to the scheme with smoothing operator

Ryc. 3. Zbieżność modelu z oparatorem wygładzania

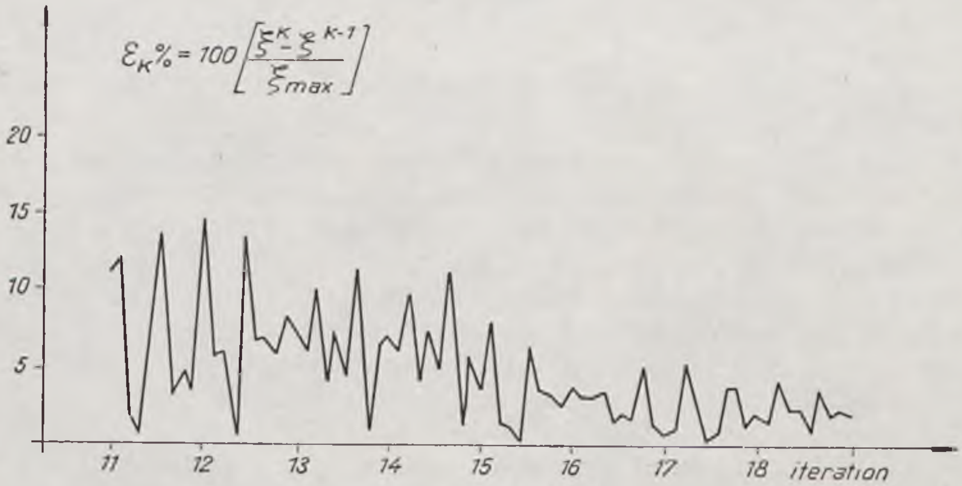

Fig. 4. Convergence relative to the scheme without smoothing operator

Ryc. 4. Zbieżność modelu bez operatora wygładzania

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