Papers

Radiance reflectance of homogeneous plane parallel layers*

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Radiance reflectance
Stratified two-flow models
Radiative transfer

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Abstract

A differential equation for the radiance reflectance is derived. It is valid for horizontally homogeneous vertically stratified layers, *i.e.* when the inherent parameters depend on the depth only. The considerations apply to any monochromatic radiation with arbitrary angular distribution; the azimuthal invariance is not required. For homogeneous layers, where the inherent parameters are depth-independent, a formula is given for the radiance reflectance in the dependence of the 'bottom' reflectance.

Some problems related to possible applications, *e.g.* when solving the 'direct' problems of evaluating the radiance of the light field in water bodies, are presented in a brief discussion.

1. Introduction

The interpretation of remote sensing results is still a fundamental question in physical oceanography. The theoretical models of the passage of radiation through sea-water are central to this problem, owing to the need for suitably justified formulas describing the dependence of the observed quantities on the parameters sought.

The basic mathematical tool for describing the light field in the water and/or the surrounding atmosphere is the appropriate radiative transfer equation containing the usual inherent parameters. For the radiance $L_P(\vec{\xi})$

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[$Wm^{-2}sr^{-2}$] related to direction represented by the element $\vec{\xi}$ of the unit sphere Ξ at point P in a region of homogeneous refraction, the equation takes the form (see *e.g.* Preisendorfer, 1961)

$$\vec{\xi}\cdot\vec{\nabla}_P L_P(\vec{\xi}) = -A_P L_P(\vec{\xi}) + B_P L_P(\vec{\xi}), \qquad (1)$$

where the gradient $\vec{\nabla}$ acts via the coordinates of point P, A_P is a multiplicator,

$$A_P L(\vec{\xi}) := a_P(\vec{\xi})L(\vec{\xi}),$$

B_P stands for the integral operator

$$B_P L(\vec{\xi}) := \int_\Xi \beta_P(\vec{\xi} \mid \vec{\eta})L(\vec{\eta})d\sigma(\vec{\eta}),$$

and σ denotes the steradian measure on Ξ. The inherent parameters used here, *i.e.* the volume absorption coefficient $a(.)$ [m^{-1}] and the volume scattering function $\beta(.\mid.)$ [$m^{-1}sr^{-1}$] and their dependence on the position of point P, are of paramount importance to the methods of describing the distribution of matter within the observed medium. Their applications have been extensively reviewed, see *e.g.* Jerlov (1976), Dera (1992). It is easily found that an important set of interesting problems is related to the solution of equation (1) for horizontally homogeneous and, possibly, vertically stratified optical layers. Such layers are usually observed while being irradiated by sunlight from above, so the radiance in the medium can be assumed to be horizontally homogeneous, too.

Thus the dependence on the position of point P reduces to the dependence on the vertical coordinate x [m] oriented upwards. In any list of coordinates it is assumed to be the first one. Consequently, for $\vec{\xi} = (\xi_1, \xi_2, \xi_1)$ in Ξ, ξ_1 stands for the variable usually denoted by $\mu = \cos\theta$ in the corresponding polar coordinates (θ, ϕ) of $\vec{\xi}$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.

According to this symmetry, (1) takes the following form,

$$\xi_1 D L_x(\vec{\xi}) = -A_x L_x(\vec{\xi}) + B_x L_x(\vec{\xi}), \qquad (2)$$

with $D = \partial/\partial x$.

The aim of this work is to contribute to the solution of equation (2). The main goal is to derive a differential equation for the radiance reflectance, and to show its importance in the 'direct' problems of determining the radiance in the whole layer, when the downwelling radiance at the upper surface and the radiance reflectance of the bottom are given together with the inherent parameters $a(.)$ and $\beta(.\mid.)$.

The assumptions made here are appropriate for an unpolarised monochromatic light field in a horizontally invariant medium and they admit the azimuthal anisotropy of all parameters (see Preisendorfer and Mobley (1984), Mobley and Preisendorfer (1988), Mobley (1989), and the references given therein). However, the structure proposed here might be competitive with the Monte Carlo methods employed by these workers owing to the possibility suggested here of evaluating the radiance reflectance (see sections 2–4). For the sake of brevity, the use of the reflectance for the radiance is outlined only (cf. section 5).

2. The radiance reflectance

To define the radiance reflectance, we have to decompose the radiance L_x at each level x into the 'two-flow' model. For every $\vec{\xi}$ in $\Xi^\uparrow$, *i.e.* $\xi_1 > 0$, we introduce the normal components $L_x^\uparrow(\vec{\xi})$ and $L_x^\downarrow(\vec{\xi})$ [$\mathrm{Wm^{-2}sr^{-2}}$] of the up- and downwelling radiances as follows:

$$L_x^\uparrow(\vec{\xi}) = \xi_1 L_x(\vec{\xi}), \qquad L_x^\downarrow(\vec{\xi}) = \xi_1 L_x(-\vec{\xi}). \tag{3}$$

In these terms, the radiance reflectance $R_x^{\uparrow\downarrow}$ of the layer below the level x for the irradiation from above is the integral operator relating the upwelling 'reflected' radiance $L_x^\uparrow$ to the downwelling 'source' radiance $L_x^\downarrow$:

$$L_x^\uparrow(\vec{\xi}) = R_x^{\uparrow\downarrow} L_x^\downarrow(\vec{\xi}) := \int_{\Xi^\uparrow} \rho_x^{\uparrow\downarrow}(\vec{\xi} \mid \vec{\eta}) L_x^\downarrow(\vec{\eta}) d\sigma(\vec{\eta}), \tag{4}$$

for arbitrary $L_x^\downarrow$, $\vec{\xi}$ in $\Xi^\uparrow$. Note, that if x denotes the level of the bottom, the equality (4) defines the radiance reflectance of the bottom surface for irradiation from above.

The radiance reflectance $R_x^{\downarrow\uparrow}$ for the layer above level x is defined in a similar manner. But then the upwelling radiance comes from the sources located at or below x and the downwelling radiance is the response of the upper layer to it in the absence of internal sources.

3. The differential equation for the radiance reflectance

Decomposition (3) allows us to rewrite equation (2) as the homogeneous linear system of two differential equations (the directional variable $\vec{\xi}$ of the positive hemisphere $\Xi^\uparrow$ is omitted whenever possible),

$$DL_x^\uparrow = -(A_x^\uparrow - B_x^{\uparrow\uparrow})L_x^\uparrow + B_x^{\uparrow\downarrow}L_x^\downarrow, \tag{5}$$

$$DL_x^\downarrow = -B_x^{\downarrow\uparrow}L_x^\uparrow + (A_x^\downarrow - B_x^{\uparrow\downarrow})L_x^\downarrow, \tag{6}$$

where D denotes the derivative with respect to the essential spatial (*i.e.* vertical) coordinate x, $D = \partial/\partial x$. Now, if compared to (1), the operators A-s and B-s are suitably restricted within the range of ξ-s and η-s, namely, for every $\vec{\xi}$ in $\Xi^\uparrow$, the following equalities

$$A_x^\uparrow L^\uparrow(\vec{\xi}) := a_x(\vec{\xi})\xi_1^{-1}L^\uparrow(\vec{\xi}),$$

$$A_x^\downarrow L^\downarrow(\vec{\xi}) := a_x(-\xi)\xi_1^{-1}L^\downarrow(\vec{\xi}),$$

$$B_x^{\uparrow\uparrow}L^\uparrow(\vec{\xi}) := \int_{\Xi^\uparrow} \beta_x(\vec{\xi} \mid \vec{\eta})\eta_1^{-1}L^\uparrow(\vec{\eta})d\sigma(\vec{\eta}),$$

$$B_x^{\uparrow\downarrow}L^{\downarrow}(\vec{\xi}) := \int_{\Xi^{\uparrow}} \beta_x(\vec{\xi}\,|\,-\vec{\eta})\eta_1^{-1}L^{\uparrow}(\vec{\eta})d\sigma(\vec{\eta}), \quad etc.$$

define the operators for arbitrary $L^{\uparrow}$ and $L^{\downarrow}$. The radiance dependences (5–6) at level x allow us to compute the result of adding a layer of thickness h, $h > 0$, $h \to 0+$ as follows:

$$\begin{aligned} R_{x+h}^{\uparrow\downarrow}L_{x+h}^{\downarrow} = L_{x+h}^{\uparrow} &\cong L_x^{\uparrow} + hDL_x^{\uparrow} \\ &\cong ((1 - hA_x^{\uparrow} + hB_x^{\uparrow\uparrow})R_x^{\uparrow\downarrow} + hB_x^{\uparrow\downarrow})L_x^{\downarrow}, \end{aligned} \tag{7}$$

$$L_{x+h}^{\downarrow} = L_x^{\downarrow} + hDL_x^{\downarrow} \cong -(hB_x^{\downarrow\uparrow}R_x^{\uparrow\downarrow} + 1 + h(A_x^{\downarrow} - B^{\downarrow\downarrow})L_x^{\downarrow}. \tag{8}$$

Therefore, the approximate changes of the radiance reflectance are given by

$$\begin{aligned} R_{x+h}^{\uparrow\downarrow} - R_x^{\uparrow\downarrow} \cong (R_{x+h}^{\uparrow\downarrow}B_x^{\downarrow\uparrow}R_x^{\uparrow\downarrow} - R_{x+h}^{\uparrow\downarrow}(A^{\downarrow} - B^{\downarrow\downarrow}) + \\ -(A^{\uparrow} - B^{\uparrow\uparrow})R_x^{\uparrow\downarrow} + B_x^{\uparrow\downarrow})h. \end{aligned}$$

This implies the differential equation

$$DR_x^{\uparrow\downarrow} = R_x^{\uparrow\downarrow}B_x^{\downarrow\uparrow}R_x^{\uparrow\downarrow} - R_x^{\uparrow\downarrow}(A_x^{\downarrow} - B_x^{\downarrow\downarrow}) - (A_x^{\uparrow} - B_x^{\uparrow\uparrow})R_x^{\uparrow\downarrow} + B_x^{\uparrow\downarrow}. \tag{9}$$

Reversing the directions in these considerations leads us to the following differential equation of the radiance reflectance for the layer above level x, irradiated from below,

$$DR_x^{\downarrow\uparrow} = -R_x^{\downarrow\uparrow}B_x^{\uparrow\downarrow}R_x^{\downarrow\uparrow} + R_x^{\downarrow\uparrow}(A_x^{\uparrow} - B_x^{\uparrow\uparrow}) + (A_x^{\downarrow} - B_x^{\downarrow\downarrow})R_x^{\downarrow\uparrow} - B_x^{\downarrow\uparrow}. \tag{10}$$

4. The solution for homogeneous layers

We start with the list of conditions essential for the considerations of this section,

- the volume absorption coefficient $a(.) = a_x(.)$ and the volume scattering kernel $\beta(.\,|\,.) = \beta_x(.\,|\,.)$ do not depend on the level x.
- the scattering of the beam has a no greater chance than all the perturbations together, *i.e.* for $\vec{\xi}$ in $\Xi^{\uparrow}$ the total volume scattering coefficient $\beta(\vec{\xi})$

$$\beta(\vec{\xi}) := \int_{\Xi} \beta(\vec{\eta}\,|\,\vec{\xi})d\sigma(d\vec{\eta}) \leq a(\vec{\xi}). \tag{11}$$

Moreover, for layers bounded from below, we put the boundary at $x = 0$ and assume the following additional requirement

- the boundary reflects no more energy than it obtains from the irradiating light, *i.e.* the radiance reflectance is represented by a sub-stochastic kernel

$$\int_{\Xi^{\uparrow}} \rho_0(\vec{\eta}\,|\,\vec{\xi})d\sigma(\vec{\eta}) \leq 1 \quad \text{for } \vec{\xi} \text{ in } \Xi^{\uparrow} \tag{12}$$

in either case, of $\rho_0 = \rho_0^{\uparrow\downarrow}$ and $\rho_0 = \rho_0^{\downarrow\uparrow}$.

In such cases, the total power leaving the domain is no greater than that entering it, *i.e.* for the case when the sources are above level x we have

$$T_x^{\uparrow} := \int_{\Xi\uparrow} L_x^{\uparrow}(\vec{\xi})d\sigma(\vec{\xi}) \quad \leq \quad T_x^{\downarrow} := \int_{\Xi\uparrow} L_x^{\downarrow}(\vec{\xi})d\sigma(\vec{\xi}).$$

Consequently, the reflectances $R_x^{\uparrow\downarrow}$ and $R_x^{\downarrow\uparrow}$ are substochastic integral operators at every level x crossing the layer.

The formula presented below serves for layers bounded from below by the bottom at $x = 0$, thus the coordinate x varies within $0 \leq x < \infty$. It depends on the reflectances of infinitely thick domains, observed from above and from below respectively. They are defined as the limits

$$R_\infty^{\uparrow\downarrow} := \lim_{x\to\infty} R_x^{\uparrow\downarrow}, \qquad R_\infty^{\downarrow\uparrow} := \lim_{x\to\infty} R_{-x}^{\downarrow\uparrow} \tag{13}$$

whenever the limits exist independently of the radiance reflectance of the bottom or top respectively. For this to hold, and also for some other steps performed below to be correct, an additional requirement should be imposed upon the inherent parameters defining A and B. The details will not be entered into. Suffice it to say that some sharpness of the inequality in (11) is quite adequate. An example of this kind says that

– for some positive $\varepsilon > 0$, common to all $\vec{\xi}$, we have

$$b(\vec{\xi}) > \varepsilon, \qquad a(\vec{\xi}) - b(\vec{\xi}) > \varepsilon. \tag{14}$$

This implies that the limits in (13) are contracting operators. Moreover, these limits are then constant solutions of the corresponding equations of (9) and (10), which yield

$$R_\infty^{\uparrow\downarrow}B^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow} - R_\infty^{\uparrow\downarrow}(A^{\downarrow} - B^{\downarrow\downarrow}) - (A^{\uparrow} - B^{\uparrow\uparrow})R_\infty^{\uparrow\downarrow} + B^{\uparrow\downarrow} = 0, \tag{15}$$

$$R_\infty^{\downarrow\uparrow}B^{\uparrow\downarrow}B_\infty^{\downarrow\uparrow} - B_\infty^{\downarrow\uparrow}(A^{\uparrow} - B^{\uparrow\uparrow}) - (A^{\downarrow} - B^{\downarrow\downarrow})R_\infty^{\downarrow\uparrow} + B^{\downarrow\uparrow} = 0. \tag{16}$$

Consequently,

$$(1 - R_\infty^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow})C^{\downarrow}R_\infty^{\downarrow\uparrow} + R_\infty^{\downarrow\uparrow}C^{\uparrow}(1 - R_\infty^{\uparrow\downarrow}R_\infty^{\downarrow\uparrow}) =$$
$$= (1 - R_\infty^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow})B^{\downarrow\uparrow}(1 - R_\infty^{\uparrow\downarrow}R_\infty^{\downarrow\uparrow})$$

where

$$C^{\downarrow} := A^{\downarrow} - B^{\downarrow\downarrow} - B^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow}, \quad C^{\uparrow} := A^{\uparrow} - B^{\downarrow\downarrow} - R_\infty^{\uparrow\downarrow}B^{\downarrow\uparrow}. \tag{17}$$

This proves that

$$S := (1 - R_\infty^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow})^{-1}R_\infty^{\downarrow\uparrow} = R_\infty^{\downarrow\uparrow}(1 - R_\infty^{\uparrow\downarrow}B_\infty^{\downarrow\uparrow})^{-1} \tag{18}$$

is the only solution of the operator equation

$$C^{\downarrow}S + SC^{\uparrow} = B^{\downarrow\uparrow}. \tag{19}$$

Knowing this, it is not hard to verify the following form of the solution of the equation (9) with constant A-s and B-s,

$$\Delta_x = \mathrm{EXP}_x^-(\Delta_0)(1 - (\mathrm{EXP}_x^+(S) - S)\mathrm{EXP}_x^-(\Delta_0))^{-1} \tag{20}$$

where one should use formula (18) for S and the notation

$$\Delta_x := R_x^{\uparrow\downarrow} - R_\infty^{\uparrow\downarrow}$$
$$\mathrm{EXP}_x^+(S) := \exp(C^\downarrow x) S \exp(C^\uparrow x),$$
$$\mathrm{EXP}_x^-(\Delta) := \exp(-C^\uparrow x)\Delta \exp(-C^\downarrow x)$$

with C-s given by (17).

It is interesting to outline the derivation of solution (20) to equation (9) for a homogeneous medium. To this end, let the integral operator Δ_x have the inverse $G_x = (R_x^{\uparrow\downarrow} - R_\infty^{\uparrow\downarrow})^{-1}$. Equation (9) implies the one below for the difference Δ_x , cf. (15–16),

$$D\Delta_x = \Delta_x B^{\downarrow\uparrow}\Delta_x - \Delta_x(A^\downarrow - B^{\downarrow\downarrow} - B^{\downarrow\uparrow}R_\infty^{\uparrow\downarrow}) +$$
$$-(A^\uparrow - B^{\uparrow\uparrow} - R_\infty^{\uparrow\downarrow}B^{\downarrow\uparrow})\Delta_x = \Delta_x B^{\downarrow\uparrow}\Delta_x - \Delta_x C^\downarrow - C^\uparrow\Delta_x.$$

Now, since $G_x\Delta_x = \Delta_x G_x = 1$, and $DG_x = -G_x D\Delta_x G_x$, a linear differential equation is arrived at for the inverses,

$$DG_x = -B^{\downarrow\uparrow} + C^\downarrow G_x + G_x C^\uparrow$$

or, according to (15), we also have

$$D(G_x - S) = C^\downarrow(G_x - S) + (G_x - S)C^\uparrow.$$

Thus, by direct inspection, the formula

$$G_x - S = \exp(C^\downarrow x)(G_0 - S)\exp(C^\uparrow x) = \mathrm{EXP}_x^+(G_0 - S)$$

is reached. Standard transformations lead to (20), since

$$\Delta_x = R_x^{\uparrow\downarrow} - R_\infty^{\uparrow\downarrow} = G_x^{-1} = (\mathrm{EXP}_x^+(\Delta_0^{-1} - S) + S)^{-1} =$$
$$= \mathrm{EXP}_x^-(\Delta_0)(1 - \exp(xC^\downarrow)S\Delta_0\exp(-xC^\downarrow) + S\,\mathrm{EXP}_x^-(\Delta_0))^{-1}.$$

It should be stressed that the simplifying assumption of the existence of the inverse G_x is not necessary for the final formula (20) to be valid, only the conditions listed are essential.

5. Comments

Knowing the dependence of the radiance reflectance at level x, the system of equations (5–6) for the two functions can be reduced to one equation for one function only, say the downwelling radiance. Indeed, by making use of the defining formula (4), the equation of (6) can be rewritten as

$$DL_x^\downarrow = (-B_x^{\downarrow\uparrow}R_x^{\uparrow\downarrow} + A_x^\downarrow - B_x^{\downarrow\uparrow})L_x^\downarrow. \tag{21}$$

Then, the upwelling radiance is the result of applying the reflectance according to (4).

Equation (21) becomes a Cauchy problem if the downwelling radiance at any level is known. The complexity is dependent on the size of the possible matrices approximating the kernels of the operations in (21). According to Mobley (1989), a suitable transformation of the problem for functions on the hemisphere into a problem for matrices can be performed by averaging over quads which form a partition of the hemispheres. This method can also be applied to equation (9) for the radiance reflectance whenever it is known at the bottom. However, in this case the unknown function of x is a square matrix, unlike the case of equation (21), where the unknown function is a column. In both cases the problem is solvable by standard procedures for first-order differential equations with a known initial (multidimensional) value.

One disadvantage should be stressed, however. Before attempting to solve (21), the data related to the reflectance should be available. Thus, the program has to be divided into two parts: first solve (9), given the radiance reflectance of the bottom, then solve (21), given the results of the first step and the downwelling radiance at the top surface. However, the Monte Carlo method used by Preisendorfer and Mobley (1986) could be replaced by this direct computation.

The scheme also applies to homogeneous layers. But should one wish to make use of the 'analytical' formulas, the following should be taken into account. Knowing the reflectance of the infinitely thick layers is essential. Fortunately, under the usual conditions, the limits (13) defining them are obtainable directly from the differential equations (9) and (10). One can start with any substochastic kernel as the bottom reflectance, say the zero kernel corresponding to a 'black' bottom. The consecutive steps of solving the coresponding difference scheme should lead to the limit, since the solution is a stable fixed-point of the problem.

One more detail is important. The transformation of the kernels into quad-average matrices should be performed very carefully. The key point here is that the averaged composition of two kernels is not equal to the product of the two matrices of averaged kernels. Special attention must be given to the factor $1/\mu$, which also requires appropriate averaging (cf. Mobley 1989).

This paper closes with the theoretical formula for the downwelling radiance in a bottomless, infinitely thick homogeneous optical layer. According to equation (21) and using the notation introduced in the preceding section

$$L_x^{\downarrow} = \exp(xC^{\downarrow})L_0^{\downarrow} \quad \text{for} \quad x < 0.$$

Exponential decay with increasing depth below level 0 is thus possible if and only if the downwelling radiance is an eigenfunction of the integral operator $C^{\downarrow}$. If it is a combination of two or more such functions, then the most slowly varying component will determine the attenuation coefficient. In order to be able to claim the existence of the lower bound for the eigenvalues, the operator should be analysed in much greater detail. Also, it is not quite clear whether the assumption made by Zaneveld (1989) with respect to the attenuation coefficients for the scalar and vector irradiances is sufficiently justified by the theory of radiative transfer.

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