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EDGE WAVES ON SHEAR FLOW

1. INTRODUCTION

Edge waves, or trapped waves play an important role in different morphological and hydrodynamic processes, which can be observed in the near-shore region. At the same time they represent very interesting solutions of the water-wave equations for a sloping beach. These are waves propagating along the beach with amplitude decaying exponentially out to sea. Ursell [13] showed that for a straight beach with a constant slope tg β, there is a sequence of edge waves, the nth mode of which has a frequency $\sigma = [gk \sin (2n+1) \beta]^{1/2}$ where $(2n+1) \beta \leqslant \frac{\pi}{2}$ and k is the wave number. Thus, the first edge-wave solution acc. to Stokes (1846) — the pioneer of the edge wave theory, is only the zero mode in this sequence. The continuous spectrum of harmonic waves and the above finite discrete spectrum give all exact solutions to the full linear wave problem on a uniformly sloping beach.

Further development of the edge-wave theory requires more assumptions. Eckart (1951) considered the same problem using the linear shallow-water approximation and found an infinite set of edge-wave modes with $\sigma = [gk\ (2n+1)\ \mathrm{tg}\beta]^{1/2}$. This solution indicates good conformity with Ursells' results for sufficiently small values of $(2n+1)\beta$ and is valid only for fairly short distances from the coast because of the failure of the shallow-water assumption in the deep water at infinity. Nevertheless, in view of the simpilcity of Eckarts' solution, his results are widely applied for practical purposes. For less idealized topographies it is impossible to obtain similar accurate sotutions.

Our knowledge of edge waves has significantly expanded in the last fifteen years. Different shallow-water procedures have been developed to describe trapped waves around islands, on the continental slopes and on other more realistic depth distributions [3], [6], [10],... Nonlinear adge-wave problems are considered in both shallow-water [7], [9] and full formulations [13], where, as can be expected, the frequency is expressed as a function of amplitude. The problem of edge-wave generation, which is one of the most interesting problems in edge-wave studies, is considered in [4], [8], [9],... It is worth mentioning here that many of the obtained results are verified by experimental works. One area of interest centred on the effects of edge waves on rhythmic features along beaches, which include beach cusps, cresentic bars and rip currents. There are numerous references in [5].

So far in edge-wave studies the absence of flow in the water is assumed. But it is well-known that in many circumstances, along beaches there are flows whose velocity changes with depth. The main aim of this work is to consider edge-waves on these shear flows. We shall restrict our attention to the linear shallow-water theory and to depth distributions bounded at infinity. In § 2 the basic equation for the surface elevation in the case of both arbitrary velocity and depth distribution is given. The possibility of edge-wave existence for a linear velocity profile and for a wide class of depth profiles is also discussed. In § 3 the so called quasi-edge waves in immobile water on a broken-line depth are shown to be the single solution which gives the continuous hydrodynamic pattern at the breaking point. These waves descrease exponentially with distance from the origin to this point as ordinary edge waves and after that keep a constant amplitude as harmonic waves. In § 4 an exact quasi-edge solution is obtained for the case of a linear velocity profile. In the last part, an approximate procedure for quasi-edge waves in the case of a polinomial velocity profile is presented and is described in detail for a parabolic velocity profile. The effects of shear flows are discussed. It is shown how to expande the procedure to a broken-line depth with more points of breaking.

2. BASIC EQUATION

Suppose that in the water body shown in Fig. 1 there is a shear flow described by

$$U^0 = U^0(z)\,, \quad V^0 = W^0 = 0\,, \quad P^0 = -\varrho g z\,, \tag{2.1}$$

We will assume that the depth contours are parallel to a straight coast line and that far from the coast the depth tends to a constant value h_1. In general, near the coast the flow must change with y also and viscosity cannot be ignored there. Assuming, however, as is usually accepted in the water wave theory, that the complicated behaviour of water motion in a small coastal region does not influence edge wave parameters significantly, we can consider (2.1) as one of the possible flows in the near shore zone.

The linearized shallow-water equations in this case can be written as follows

$$\vec{u}_t + U^0 \vec{u}_x + w U^0_z \vec{i}_1 = -\operatorname{grad} p/\varrho$$

$$p_z = 0 \,, \quad \operatorname{div} \vec{u} = 0 \tag{2.2}$$

where $\vec{u}$ (u, v, w) and p denote velocity (with ist components) and pressure associated with wave motion, $\vec{i}_1$ is the unit vector in the X direction. Let $\xi(x, y, t)$ denote the surface elevation and $z = -H(y)$ the sea depth, then boundary conditions are

$$\xi_t + U \xi_x = w \qquad \text{at } z = 0 \tag{2.3}$$

$$p - \varrho g \xi = 0 \qquad \text{at } z = 0 \tag{2.4}$$

$$v \cos \Theta + w \sin \Theta = 0 \qquad \text{at } z = -H(y) \tag{2.5}$$

$$\xi \text{ must be bounded both at } y = 0 \text{ and } y = \infty \tag{2.6}$$

Solutions which represent edge waves travelling in the flow-direction with the wave number k and frequency σ are found in the form

$$\{u, v, w, \xi\} = \{U(y,z), V(y,z), W(y,z), a(y)\} \exp i(kx - \sigma t) \tag{2.7}$$

As we are concerned here only with long waves, the function $f = \frac{\sigma}{k} - U^0(z)$ can be assumed to be positive everywhere for $-H(y) \leq z \leq 0$. Substituting (2.7) into (2.2) and using (2.4), (2.5) with the obvious relation: $\operatorname{cotg} \Theta = \operatorname{tg} \beta = H'(y)$, where Θ and β are shown in Fig. 1, we obtain the following equations:

$$U = i(V_y + W_z)/k \,, \quad V = -iga'/(kf) \,, \quad P = \varrho g \xi$$

$$W = \frac{ig}{k}(a'' - k^2 a) f \int_{-H(y)}^{z} f^{-2}(z)\, dz + iga'H'/(kf)\Big|_{z=-H} \tag{2.8}$$

From (2.3) and (2.8) the equation for the amplitude $a(y)$ is

$$F_1 F_2 a'' + H' a' + k^2 F_2 \left(\frac{1}{g} - F_1\right) a = 0 \tag{2.9}$$

where
$$F_1 = \int_{-H}^{0} f^{-2} dz \,, \quad F_2 = f\Big|_{z=0} \cdot f\Big|_{z=-H} \tag{2.10}$$

or in the self-adjoint form:

$$[E_1(y,c_e)a']' + E_2(y,c_e)\,a = 0\,, \quad c_e = \sigma/k \tag{2.11}$$

where
$$E_1 = \exp\int_0^y H'F_1^{-1}F_2^{-1}dy\,, \quad E_2 = k^2 E_1(g^{-1}F_1^{-1} - 1) \tag{2.12}$$

The prims denote derivation with respect to y. In the framework of shallow-water theory (2.9) is the basic equation in edge wave studies for the case of an arbitrary shear-flow velocity profile and an arbitrary depth distribution.

It is interesting to note that, of the sea depth is constant and a is found in the form of sinusoidal waves: $a = a_0 \exp(iky)$, then (2.9) reduces to the well-known Burns' dispersion relation for long waves on shear flow [2]:

$$g\int_{-H}^{0}[c_e - U^0(z)]^{-2}dz = 1$$

as can be expected. On the other hand, if there is no shear flow, (2.11) yields

$$(Ha')' + \left(\frac{\sigma^2}{g} - k^2H\right)a = 0 \tag{2.13}$$

This is the shallow-water wave equation in immobile water for an arbitrary depth distribution. Eq. (2.9) and the condition obtained from (2.6):

$$a \text{ is bounded both at } 0 \text{ and at } \infty \tag{2.14}$$

formulate an eigenvalue problem which can, in general, be solved only by numerical or approximate methods (even the much simpler Eq. (2.13) has no exact solutions). It is worth considering the nature of the spectrum of the formulated problem (2.11) and (2.14). Unfortunately, the relevant theory, as can be seen from the lastest achivements in boundary problems of differential equations, is not yet sufficiently developed. However, if we restrict ourselves to the case of a linear velocity profile:

$$U^0(z) = U_0 + U_1 z \tag{2.15}$$

we can show some properties of the spectrum for a large class of depth distributions which decribe quite realistic seabed profiles:

$$H = y\,tg\beta \quad as \quad y \to 0, \quad H = h_1 \quad as \quad y \to \infty$$
$$H' \geq 0, \quad H'' \leq 0, \quad H''' \geq 0 \tag{2.16}$$

One of the suitable depth distributions, for example, is $h_1(1-e^{-y\,tg\beta/h_1})$. Taking into account (2.15) we obtain from (2.11) and (2.12)

$$(Ha')' + (\omega^2/g - K^2 H)a = 0 \tag{2.17}$$

where $\omega = \sigma - kU_0, \quad K^2 = k^2(1 - \omega U_1/gk)$ (2.18)

In order to consider the spectrum of (2.17) and (2.14), it is convenient to reduce (2.17) to the normal from applying the Liouville transformation (see [12])

$$u(s) = H^{1/4}a \qquad s = \int_0^y H^{-1/2}dy$$

Then (2.17) can be written as

$$u'' + (\lambda - q)u = 0 \tag{2.19}$$

where $\lambda = \omega^2/g, \qquad q = K^2 H - \frac{1}{16}\frac{H'^2}{H} + \frac{1}{4}H''$ (2.20)

Invoking the relations $s = 2\,(y/tg\,\beta)^{1/2}$ as $y \to 0$, $\frac{dy}{ds} = H^{1/2} \geq 0$ and (2.16) we can show that q increases with s for $0 \leq s < \infty$, $q \approx (s^2K^2tg^2\beta - s^{-2})/4$ for $s \to 0$ and $q \to K^2h_1$ for $s \to \infty$. From the relevant theorems in [11] we know that the singularity at $s = 0$ in this case allows us to consider the problem as if there were no singularity at all. The spectrum is discrete in the interval $\lambda < K^2h_1$ and continuous in ahe interval $\lambda > K^2h_1$. The first interval corresponds to edge waves, while the second one to ordinary harmonic waves. The number of discrete modes can be infinite or finite depending on whether the point $\lambda = K^2h_1$ is a point of accumulation or not. It is interesting to note that the set of point eigenvalues, therefore, can even be empty. The results obtained in the next parts confirm the possibility of absence of edge waves, although the function $H(y)$ there will be assumed to have discontinuous first derivative at some point, say, l_1.

$$H(y) = \begin{cases} y\,tg\beta & \text{for} \quad 0 \leq y \leq l_1 \\ h_1 = l_1\,tg\beta & \text{for} \quad l_1 \leq y < \infty \end{cases} \tag{2.21}$$

It will be shown that for the depth profile (2.21) edge waves, in the classical sense, do not exist, but it is possible to have so called „quasi-edge waves".

3. QUASI-EDGE WAVES IN IMMOBILE WATER

In order to consider quasi-edge waves on shear flow, it is useful to study these waves first for the case of absence of any shear flow. As mentioned in the end of § 2 we will only be concerned with the depth distribution described by (2.21). The equation (2.13) then reduces to

$$y a'' + a' + (\sigma^2/g\,\mathrm{tg}\beta - k^2 y)\, a = 0 \qquad \text{for} \quad 0 \leq y \leq l_1 \qquad (3.1)$$

$$a'' + (\sigma^2/g h_1 - k^2) a = 0 \qquad \text{for} \quad l_1 \leq y < \infty \qquad (3.2)$$

At $y = l_1$ the hydrodynamic pattern must be continuous. It demands continuity of the surface elevation and its slope, i.e.

$$[a] = 0 \quad , \quad [a'] = 0 \qquad \text{at} \quad y = l_1 \qquad (3.3)$$

Solutions to (3.1) can be found in the form

$$a = e^{-ky}\,\varphi(y) \qquad (3.4)$$

where $\varphi(y_1)$ must satisfy the Laguerre equation

$$y_1 \varphi_{y_1 y_1} + (1 - y_1)\varphi_{y_1} + \lambda\varphi = 0 \quad , \quad y_1 = 2ky, \qquad (3.5)$$

$$\lambda = (\sigma^2/g\,\mathrm{tg}\beta - k)/2k$$

The condition (2.14) requires the regularity of φ in the vicinity of $y_1 = 0$, while the first condition in (3.3) involves the boundedness of φ at l_1. From the Laguerre equation theory [12] we know that all these conditions will be satisfied by

$$\varphi(y_1) = \sum_{\nu=0}^{\infty} c_\nu y_1^\nu \qquad (3.6)$$

Indeed, substituting (3.5) into (3.4) yields

$$c_{\nu+1} = \frac{\nu - \lambda}{(\nu+1)^2}\, c_\nu \quad , \quad c_\nu = (-1)^\nu \frac{\lambda(\lambda-1)\ldots(\lambda-\nu+1)}{(\nu!)^2}\, c_0 \qquad (3.7)$$

The series (3.6) has infinite radius of convergence. Hance φ at l_1 is bounded and is equal to

Let us consider now Eq. (3.2). Notice that the shallow-water wave equation in the interval $[l_1, \infty]$ is

$$\xi_{xx} + \xi_{yy} - \frac{1}{gh_1}\xi_{tt} = 0 \qquad (3.8)$$

If ξ is a simple harmonic wave, i.e. $\xi = \exp i(kx + my - \sigma t)$ (3.8) involes $\sigma^2/gh_1 = k^2 + m^2$/ Hence the phase velocity $c = \sigma/(k^2 + m^2)^{1/2} = (gh_1)^{1/2}$. It means that $\sigma^2/gh_1 - k^2$ in (3.2) may be positive and Eq. (3.2) can have oscillatory solutions

$$a = A\cos my + B\sin my \qquad (3.9)$$

$$m^2 = \frac{\sigma^2}{gh_1} - k^2 \qquad (3.10)$$

Using (3.3) we obtain

$$A = D_1\cos ml_1 - D_2\sin ml_1, \quad B = D_1\sin ml_1 + D_2\cos ml_1$$

$$D_1 = e^{-kl_1}\varphi(l_1), \quad D_2 = e^{-kl_1}[\varphi'(l_1) - k\varphi(l_1)]$$

$$\varphi'(l_1) = 2k\sum_{\nu=1}^{\infty}\nu c_\nu(zkl_1)^{\nu-1} \qquad (3.11)$$

In view of (3.5), (3.9) can be written as

$$a = (D_1^2 + D_2^2)^{1/2}\cos[m(y - l_1) - \delta], \quad \delta = \operatorname{arc\,tg}(D_2/D_1) \qquad (3.12)$$

Following Longuet-Higgins [6] we represent ξ in the form

$$\xi = \frac{1}{2}(D_1^2 + D_2^2)^{1/2}\{\exp i[kx - m(y - l_1) - \sigma t - \delta] + \exp i[kx + m(y - l_1) - \sigma t + \delta]\} \qquad (3.13)$$

The first term in (3.13) describes a wave propagated from infinity to the right of the discontinuity while the second one can be considered to be the reflected wave propagated to infinity. For the angle α of incidence at $y = l_1$, $\sin\alpha = k/(m^2 + k^2)^{1/2} = k(gh_1)^{1/2}/\sigma$ and there the wave undergoes a phase-change of 2δ It is important to emphasize that non-oscillatory solutions do not exist, as the conditions (3.3) can not be satisfied by non-oscillatory solutions. So far λ in (3.5) can have every real value unless any additional condition for the behaviour of φ is made at $y = l_1$. Remembering that we want to have a wave decreasing exponentially in the interval $[0, l_1]$ then we must assume that φ may increase at most algebraically at l_1, i.e. there must exist a positive integer N and a positive constant A° such that

$$|\varphi(l_1)| < A^0 l_1^N \tag{3.14}$$

Hence the problem under consideration has a solution if and only if λ has one of the values 0, 1, 2, 3, ..., n, because the series (3.6) reduces then to the Laguerre polinomials of degree n/ then $c_j = 0$ for $j>n$/. This means

$$\lambda = \frac{1}{2k}\left(\frac{\sigma^2}{g\,tg\beta} - k\right) = n \quad \text{or} \quad \sigma^2 = gk(2n+1)\,tg\,\beta$$

$$\varphi(y) = L_n(2ky), \quad L_n(\xi) = \frac{e^{\xi}}{n!}\frac{d^n}{d\xi^n}(e^{-\xi}\xi^n) \tag{3.15}$$

$$L_0 = 1, \quad L_1 = 1 - \xi, \quad L_2 = 1 - 2\xi + \frac{1}{2}\xi^2,$$

$$L_3 = 1 - 3\xi + \frac{3}{2}\xi^2 - \frac{1}{6}\xi^3 \ldots$$

The obtained solution presented by (3.4) and (3.15) for $0 \leqslant y \leqslant l_1$ as well as by (3.9) and (3.10) for $l_1 \leqslant y \leqslant \infty$ describes a wave which exponentially decreases with y for $0 \leqslant y \leqslant l_1$ and has a constant amplitude for $l_1 \leqslant y < \infty$. If l_1 is large enough the amplitude for $y \geqslant l_1$ is quite small and the wave is, as may be said, trapped at the shore. Waves of this type will be called „quasi-edge waves". They are exact solutions of the formulated eigevalue problem when the depth profile is the broken line (2.21). Both edge waves and quasi-edge waves have the same modes defined by the relation $\sigma^2 = gk(2n+1)\,tg\,\beta$ and they are identical for $0 \leqslant y \leqslant l_1$, but they are quite different in the second interval because edge waves continuously decrease exponentially with y. It is worth noting that while the beach slope β must be very small for edge waves (see [13]), it may, theoretically, be large enough for quasi-edge waves satisfying the inequality $0 \leqslant \beta \leqslant \pi/2$ as for the validity of (3.1) and (3.2) it is necessary only to have $kh_1 \ll 1$.

The wave pattern is schematically shown for the lowest modes (n = 0, 1, 2) in Fig. 2. The exact behaviour of the wave is given graphically, for example in [5].

4. QUASI-EDGE WAVES FOR LINEAR VELOCITY PROFILE

In this part we consider quasi-edge waves on the surface of the flow, the velocity profile of which is described by (2.15) for the depth distribution presented by (2.21). Eq.(2.17) reduces to

$$y a'' + a' + \left(\frac{\omega^2}{g\,tg\beta} - K^2 y\right) a = 0 \qquad \text{for} \quad 0 \leq y \leq l_1 \qquad (4.1)$$

$$a'' + \left(\frac{\omega^2}{g h_1} - K^2\right) a = 0 \qquad \text{for} \quad l_1 < y \leq \infty \qquad (4.2)$$

where ω and K are defined by (2.18). Using the solution of the similar problem in § 3 we can easily obtain

$$\xi = e^{i(kx-\sigma t)} \cdot \begin{cases} e^{-Kl}\, L\,(2ky) & \text{for } 0 \leq y \leq l_1 \\ (D_1^2 + D_2^2)^{1/2} \cos[m(y-l_1) - \delta] & \text{for } l_1 \leq y < \infty \end{cases} \qquad (4.3)$$

where δ is determined in (3.12), $\omega^2 = gK(2n+1)\,tg\,\beta$

$$D_1 = \exp(-Kl_1)\, L_n(2kl_1)$$
$$D_2 = K\exp(-Kl_1)\,[2L_n'(2Ky) - L_n(2Ky)]_{y=l_1} \qquad (4.4)$$

$$(\sigma - kU_0)^2 = gk(2n+1)\,[1 - U_1(\sigma - kU_0)/gk]^{1/2} tg\beta \qquad (4.5)$$

$$(\sigma - kU_0)^2 = (k^2 + m^2)\, g h_1 \qquad (4.6)$$

$$\sigma < (g + U_0 U_1)\, k / U_1 \qquad (4.7)$$

For the 0th and the 1st order we have respectively

$$D_1 = \exp(-Kl_1)\,, \quad D_2 = K\exp(-Kl_1)/m\,, \quad \delta = -\,\text{arc tg}(K/m)$$

and

$$D_1 = \exp(-Kl_1)(1 - 2Kl_1)\,, \quad D_2 = K(2Kl_1 - 3)\exp(-Kl_1)/m$$

$$\delta = \text{arc tg}\,[K(2Kl_1 - 3)/m(1 - 2Kl_1)]$$

The surface pattern is indentical with the one shown in Fig. 2. In order to show the influence of flow parameters on quasi-edge waves, it is more convenient to consider the phase velocity $c_e = \sigma/k$ and the relative phase velocity $\tilde{c}_e = c_e - U_0$. In terms of c_e and c_e from (4.5) and (4.7) we can write

$$\tilde{c}_e = (2n+1)\, g\, tg\beta (1 - \tilde{c}_e U_1/g)^{1/2}/k \qquad \text{and} \qquad \tilde{c}_e < g/U_1 \qquad (4.8)$$

$$\tilde{c}_e^2 / c_{e0}^2 = (1 - c_e U_1/g)^{1/2} \qquad c_{e0}^2 = (2n+1)\, g\, tg\beta/k \qquad (4.9)$$

It can be seen from (4.9) that the relative phase velocity of quasi--edge waves propagated in the flow-direction is smaller than the phase velocity of edge waves. Considering $\tilde{c}_e$ as a function of U_1 we can conclude that the phase velocity decreases with the gradient of shear flow. The maximal value of c_e is reached at $U_1 = 0$ and is equal to c_{eo}. So, the relation $c_e \leqslant c_{eo} + U_o$ becsmes an unequality only for the uniform flow. In addition, we can see from (4.3) that quasi-edge waves on the flow (2.15) decrease with y slower than the corresponding edge waves. The power K varies with U_o and U_1 after (2.18). The above dependence of quasi-edge waves on the flow (2.15) gives us an approximation of dependence of these waves on shear flows in general because (2.15) describes the first approximation to any real shear flow.

5. QUASI-EDGE WAVES FOR PARABOILIC VELOCITY PROFILE

It is seen above that with the exception of the linear velocity profile (2.15) exact solutions to the problem (2.9) and (2.14) are impossible. On the other hand, every velocity profile can be presented by a polinomial of suitable degree, say m, with the required accuracy. Therefore it is worth developing here a procedure of approximated solutions for a polinomial velocity profile. In this part, as an example of this procedure, approximated quasi-edge waves will be found for the parabolic profile:

$$U^0(z) = U_0(h_1^2 - z^2)/h_1^2 \tag{5.1}$$

while the depth distribution will be given as before by (2.21).

For long waves, in natural circumstances we can assume that max $U^o(z)\ /c_e \ll 1$ and $\varepsilon = \max U^o(z)/\sqrt{gh_1} \ll 1$. In our case max $U^o = U_o$ and $\varepsilon = U_o/\sqrt{gh_1}$. We can write.

$$F_1 = \frac{1}{c_e^2}\int_{-H(y)}^{0}[1 - U^0/c_e]^{-2}dz = \sum_{n=0}^{\infty}\frac{n+1}{c_e^{n+2}}\int_{-H(y)}^{0}[U^0(z)]^n dz \tag{5.2}$$

In view of (5.1) for $0 \leqslant y \leqslant l_1$ to third order of $1/c_e$ we obtain

$$F_1 = c_e^{-3} g\, tg\beta\, [c_e + 2\varepsilon\sqrt{gh_1}\,(1 - y^2 tg^2\beta/3h_1^2)] + O(c_e^{-3}) \tag{5.3}$$

hence F_1 at $y = l_1$, equals

$$F_1^* = h_1 \cdot c_e^{-3}[c_e + 4e\sqrt{gh_1}\,/3] \tag{5.4}$$

Eq. (2.9) reduces to

$$F_1 F_2 a'' + tg\beta\, a' + k^2 F_2 (\frac{1}{g} - F_1) a = 0 \quad \text{for} \quad 0 \leqslant y \leqslant l_1 \tag{5.5}$$

$$F_1^* a'' + k^2(\frac{1}{g} - F_1^*)\, a = 0 \qquad \text{for} \quad l_1 \leqslant y < \infty. \tag{5.6}$$

The condition at $y = l_1$ is (3.3)

First let us consider Eq. (5.5) Its solution can be expressed as a power series in ε.

$$a = a_0(y) + \varepsilon a_1(y) + \varepsilon^2 a_2(y) + \dots \tag{5.7}$$

$$c_e = c_{e0} + \varepsilon c_1 + \varepsilon^2 c_2^2 + \dots \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.5) and equating powers of ε we obtain

at zero order $$y a_0'' + a_0' + k^2\left(\frac{c_{e0}^2}{g\,tg\beta} - y\right) a_0 = 0 \tag{5.9}$$

at nth order $$y a_n'' + a_n' + k^2\left(\frac{c_{e0}^2}{g\,tg\beta} - y\right) a_n = \exp(-ky)\, R_{(n)}(y, c_n) \tag{5.10}$$

where in view of (5.2) R_n is a polinomial with c_n contained in the free term. The solution of (5.9) gives classical edge waves in absence of any flow as can be expected:

$$a_0 = \exp(-ky) L_n(2ky) \tag{5.11}$$

$$c_{e0}^2 = (2n+1)\, g\, tg\beta / k \tag{5.12}$$

Solutions to (5.10) can be found in the form

$$a_n = \exp(-ky)\, A_{(n)}(y) \tag{5.13}$$

where $$y A_{(n)}'' + (1 - 2ky) A_{(n)}' = R_{(n)} = R_0(c_n) + \sum_{m=1}^{N(n)} R_m y^m \tag{5.14}$$

The required integral of (5.10) is

$$a_n = \exp(-ky) \int_0^y \frac{A_0 c_n}{y}\, dy + \sum_{m=1}^{N(n)} \frac{A_m}{m} y^m \tag{5.15}$$

where $$A_N = R_N / 2k\,, \quad A_m = [(m+1) A_{m+1} - R_m] / 2k \tag{5.16}$$

$$m = 0, 1, 2, 3, \dots, N-1$$

Boundedness of a in (5.7) involves boundedness of each a_n. Therefore from (5.15) we must have an equation for c_n:

$$A_0(c_n) = 0 \tag{5.17}$$

It is interesting to note that c_n can also be evaluated from the certain orthogonality relation between exp (—ky). R_n and a_o considering a_n and a_o as functions in the interval [0, ∞] and requiring boudendness of a_n and a'_n at 0 and ∞:

$$\int_0^\infty a_0 \exp(-ky) R_{(n)}(y) dy = \{ y [a_0 a'_n - a_0 a'_n] \}_0^\infty = 0 \qquad (5.18)$$

Following, the above procedure we can obtain succesive approximations. In view of complicated final formulae we give only the first and second approximation and only for the lowest mode (n = 0). At first order

$$R_{(1)} = -k/c_{e0}[2c_1 - \frac{\sqrt{gh_1}}{h_1^2}(2h_1^2 - y^2 tg^2\beta)]$$

$$c_1 = \sqrt{gh_1}(1-R), \quad a_1 = \frac{\sqrt{gh_1}\, tg^2\beta}{2c_{e0}h_1^2} y(\frac{1}{k} + \frac{y}{2}) e^{-ky} \qquad (5.19), (5.20)$$

At second order

$$R_{(2)} = \sum_{n=0}^{5} \tilde{D}_n y^n, \quad \tilde{D}_n = -g D_n / c_{e0}^2, \quad c_{e0}^2 = g\, tg\beta / k$$

$$D_0 = kh_1(R^2 + 2c_2/\gamma\sqrt{gh_1}), \qquad D_1 = -4k^2 h_1 R^2,$$

$$D_2 = k\, tg^4\beta[4 - 5R - 4\gamma(1-R)(2-R)]/2h_1,$$

$$D_3 = tg^2\beta[k^2 + \frac{5}{24}\frac{tg^2\beta}{h_1^2} - \frac{1}{2}k^2\gamma(1-R)(2-R)]/h_1,$$

$$D_4 = k\, tg^4\beta[\gamma(1-R) - \frac{5}{4}]/3h_1^3,$$

$$D_5 = k^2 tg^4\beta[\gamma(1-R) - 1]/6h_1^3,$$

$$R = tg^2\beta / 4k^2 h_1^2, \qquad \gamma = \sqrt{gh_1}/c_{e0}$$

$$c_2 = c_{e0}R\{\gamma[R(47R - 69) + 22] + 47R - 14\} \qquad (5.21)$$

$$a_2 = \exp(-ky)\sum_{m=1}^{5} \frac{E_m}{m} y^m, \quad E_5 = -\frac{\tilde{D}_5}{2k},$$

$$E_m = \frac{(m+1)E_{m+1} - \tilde{D}_m}{2k} \qquad (5.22)$$

To second order using (5.12), (5.19)÷(5.22) we obtain

$$a = \exp(-ky)\left\{1 + \frac{U_0 \operatorname{tg}^2\beta}{2c_{e0}h_1^2}\, y\left(\frac{1}{k} + \frac{y}{2}\right) + \frac{U_0^2}{gh_1}\sum_{m=1}^{5}\frac{E_m}{m}y^m\right\} \quad \text{for} \quad 0 \leq y \leq l_1 \tag{5.23}$$

$$c_e = c_{e0} + U_0(1-R) + c_{e0}RU_0^2\{\gamma[R(47R-69)+22]+47R-14\}/4gh_1 \tag{5.24}$$

Now we return to Eq. (5.6). By analogy with the sign of the coefficient $(\sigma^2/gh_1 - k^2)$ in (3.2) we can write

$$gF_1^* = \int^{0} g\,(c_e - U^0)^{-2}\, dz \leq gh_1/(c_e - U^0_{max}) < 1$$

The solution to (5.6) satisfying (3.3) is

$$\eta = (Q_1^2 + Q_2^2)^{1/2}\cos[m(y-l_1) - \delta] \quad \text{for} \quad l_1 < y < \infty \tag{5.25}$$

where $m = k\,(1/gF_1^* - 1)^{1/2}$ and c_e in F_1^* is replaced by c_e from (5.24).

$$Q_1 = \exp(-kl_1)\left[1 + \frac{U_0}{2c_{e0}}\left(\frac{1}{2} + \frac{1}{kl_1}\right) + \frac{U_0^2}{gh_1}\sum_{m=1}^{5}\frac{E_m}{m}l_1^m\right] \tag{5.26}$$

$$Q_2 = -kQ_1 + \exp(-kl_1)\left[\left(\frac{1}{k} + \frac{l_1}{2}\right)\frac{U_0}{2c_{e0}l_1^2} + \frac{U_0^2}{gh_1}\sum_{m=1}^{5}E_m l^{m-1}\right] \tag{5.27}$$

The expressions (5.23)—(5.25) therefore give the required solution with an error of $O(U_0^2/c_{e0}^3\, gh_1)$.

The effect of the flow (5.1) on quasi-edge waves can be seen in the last two terms in each expressions (5.23), (5.24), (5.26) and (5.27).

To conclude this part, it should be noted that the procedure described above can easily be expanded to a broken line presenting the depth profile with more points of discontinuity, say, l_1, l_2, ..., l_m. It is necessary then to use also the other solution of the Laguerre equation, which is not bounded at $y = 0$ but is finite for $l_1 \leqslant y \leqslant l_j$, $1 < j \leqslant m$.

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FALE KRAŃCOWE NA POWIERZCHNI PRZEPŁYWÓW WIROWYCH

Streszczenie

Celem głównym niniejszej pracy jest rozważenie fal krańcowych na powierzchni swobodnej przepływów wirowych (*shear flows*) dla wypadku liniowej teorii płytkiej wody. Otrzymano równanie dla wzniesienia swobodnej powierzchni przy dowolnym rozkładzie prędkości i dowolnym profilu dna. Wykazano możliwość istnienia fal brzegowych dla liniowego rozkładu prędkości i szerokiej klasy profili dna. Aproksymując profil dna linią łamaną rozwiązano zagadnienie dla spokojnej wody uzyskując, jako jedyne rozwiązanie zapewniające ciągłość hydrodynamiczną, tak zwane fale quasi-brzegowe. W wypadku fal quasi-brzegowych przedstawiono dwa warianty: ścisłe rozwiązanie dla liniowego rozkładu prędkości oraz procedurę przybliżonego rozwiązania dla wielomianowego rozkładu prędkości.

Omówiono także niektóre efekty przepływów wirowych.

REFERENCES

1. Bowen, A. J. and Guza, R. T. Edge waves and surt beat. J. Geophys. Res., 1978, vol. 83, No 4.
2. Burns, J. C. Long waves in running water. Proo. Camb. Phil. Soc. 1953, vol. 49, pp. 695—706.
3. Grimshaw, R. Edge waves: A long-wave theory for oceans of finite depth. J.Fluid Mech. 1974, vol. 62, part 4.
4. Guza, R. T. and Davis, R. E. Excitation of edge waves by waves incident on a beach. J. Geophys. Res., 1974, vol. 70, No 9.
5. Guza, R. T. and Inman, L. D. Edge waves and beach cusps. J. Geophys. Res., 1975, vol. 80, No 21.
6. Longuet-Higgins, M.S. On the trapping of wave energy round islands. J. Fluid Mech., 1967, vol. 29, part 4.
7. Minzoni, A. A. Nonlinear edge waves and shallow-water theory. J. Fluid Mech., 1976, vol. 74, part 2.
8. Minzoni, A. A. and Whitham, G. B. On the excitation of edge waves on beaches. J. Fluid Mech., 1977, vol. 79, part 2.
9. Rocliff, N. Finite amplitude effects in free and forced edge waves. Math. Proc. Camb. Phil. Soc., 1978, vol. 83, pp. 463—479.
10. Shen, M. C. and Meyer, R.E. and Keller, J. B. Spectra of water waves in channels and around islands. Phys. Fluids 1968, vol. 11. No 1.
11. Titchmarsh, E.C. Eigenfunction Expantions Associated with second order Differential Equations. Part 1, 2nd. edn. Oxford University Press. 1962.
12. Tricomi, F. G. Differential Equations. Blackie and Son Limited, 1961.
13. Ursell, F. Edge waves on a sloping. Proc. Roy. Soc., 1952, A 214, pp.79-97.
14 Whitham, G. B. Nonlinear effects in edge waves. J. Fluid Mech., 1976, vol. 74, part 2.