On the transformation of long gravity waves on a sloping beach*

OCEANOLOGIA, 52 (3), 2010.
pp. 363–389.

KEYWORDS
Long wave
Wave transformation
Sloping beach
Wave run-up

Jan Kazimierz Szmidt*
Benedykt Hedzielski

Institute of Hydro-Engineering,
Polish Academy of Sciences,
Waryńskiego 17, PL–71–310 Szczecin, Poland;

e-mail: jks@ibwpan.gda.pl

*corresponding author

Received 1 July 2009, revised 31 May 2010, accepted 2 June 2010.

Abstract

The transformation of long water waves arriving at a sloping beach is investigated. An approximate theory is presented for plane periodic waves propagating in water of non-uniform depth. The theoretical description of the phenomenon, based on certain kinematic assumptions, is formulated in the material variables, and the solution is constructed by applying the Hamilton variational principle. In order to assess the accuracy of the formulation and to learn more about long wave transformation, experimental measurements were carried out in a laboratory flume. In the experiments, a water wave, generated by a piston-type wave maker placed at one end of the flume, propagated towards a rigid inclined ramp installed at the other end of the flume. The wave transformation along the direction of its propagation was recorded by a set of wave gauges installed along the flume. The wave run-up on the sloping beach was measured with a special conductivity gauge placed alongside the ramp. Comparison of the theoretical results with experimental data indicates that the proposed theoretical formulation provides a good description of

* This work was supported by the Polish Ministry of Science and Higher Education under grant No. N N506 378134.

the main features of wave transformation behaviour over a sloping beach, except in the vicinity of the shore point, where some discrepancies occur.

1. Introduction

Water waves arriving from the sea at a sloping beach undergo changes as a result of the diminishing water depth. Waves usually increase in amplitude as they approach a beach; thus, in the theoretical description of the phenomenon, wave height should be treated as finite (not infinitesimally small). At the same time, the lengths of arriving waves are large compared to the water depth; therefore, in the description of such waves, it is justified to assume that the water flow is nearly horizontal. For this reason, only the horizontal coordinates are chosen as independent variables in the description of this problem. Commonly, spatial (Eulerian) coordinates are used as independent variables in the description of these waves. In this Eulerian description, however, it is difficult to solve the boundary conditions, especially at the moving boundaries of the fluid domain. Another possibility, employed in the present work, is to use material (Lagrangian) coordinates as independent variables, since this provides a much easier solution of the boundary conditions. But this comes at a cost: the structure of the equations describing the fluid motion is more complicated.

The subject literature is considerable. A detailed discussion of long water waves may be found in Dingemans' monograph (1997), which also contains a vast bibliography on the subject; among other contributions, the one by Carrier & Greenspan (1958) stands out. On the basis of shallow water approximations, these authors discovered a hodograph transformation that allowed them to transform the original non-linear shallow-water differential equations, defined in physical space, into a single, second-order linear differential equation for a potential function defined in transformation space. This linear equation enabled them to calculate the run-up height of a non-breaking long wave of small amplitude on a sloping beach. However, with the non-linear hodograph transformation, it is difficult to specify initial or boundary data on a sloping beach for the general case (Synolakis 1987, Kânoğlu 2004). The difficulty emerges in the transformation from the physical space of a given initial condition (wave profile) into the transformation space. In order to overcome this difficulty, Synolakis (1987) used a linearized hodograph transformation with which the initial waveform in the transformation space can be defined. The detailed discussion of the problem was confined to a solitary wave, for which a formula describing the maximum run-up of the wave was derived. An approach similar to that by Synolakis is given in Kânoğlu (2004), where, in order to solve the initial value problem, the transformation is linearized in the transformation space at time $t = 0$, after which the full non-linear transformation is used to solve the initial value problem of the non-linear shallow water wave equations. In particular, Kânoğlu considered the solitary wave and $N-$ wave initial conditions. In constructing a solution to the non-linear shallow water waves, Pelinovsky (1991) introduced the similarity parameter Br: this is of fundamental importance. It was shown that for $Br < 1$ the solution is continuous and the wave runs up the beach without breaking. But when $Br > 1$, the wave breaks. Formulae for a monochromatic wave were derived, which describe the maximum run-up height and its velocity. Massel & Pelinovsky (2001) investigated the transformation of dispersive breaking waves approaching a sandy beach. On the basis of mild-slope equations in a deeper area and linear equations of shallow water in the area close to the shoreline and taking into account the dissipation of energy due to wave breaking, the authors proposed a solution to the more complex problem of the run-up of dispersive breaking and non-breaking waves. They derived formulae describing the maximum and minimum run-up distances. The good accuracy of the model's predictions was demonstrated by comparison of its results with literature data. In order to simplify Carrier & Greenspan's description of waves climbing a sloping beach, Shuto (1967) applied material coordinates as independent variables. In this material formulation, run-up was described by a linear equation obtained without any transformation. This equation was derived by means of a power series expansion of all dependent variables in the problem with respect to a small parameter. The first-order approximation yielded equations similar to those of the linear theory described in the spatial coordinates. Theoretical results were compared with experimental data. A non-linear set of equations describing long waves in the material variables was derived by Goto (1979). The equations were obtained with the help of a perturbation scheme with respect to displacements of fluid particles from their initial positions. Numerical solutions of the equations for wave run-up were compared with solutions based on the linear theory.

Another formulation of the propagation of non-linear long water waves over uneven bottoms was given by Miles & Salmon (1985). This was based on the kinematic assumption that fluid motion is 'columnar', i.e. the horizontal displacements of fluid particles forming a vertical column are the same for all particles. More recently, Wilde & Chybicki (2004) derived equations for a non-linear long wave propagating in a fluid of constant depth with the aid of the kinematic assumption that a material vertical line remains vertical during the entire motion of the fluid. Their assumption is similar to that of Miles & Salmon. A generalization of the formulation for long waves propagating in a fluid of non-uniform depth may be found in Chybicki (2006) and Szmidt (2006). In both papers, the governing equations for long waves were derived by means of a variational formulation with appropriate Lagrangian density functions. In the first paper, an approximated form of the Lagrangian density function in the action integral was used. In the second paper, the full form of the Lagrangian density function was applied, but some simplifications were introduced during the derivation of the final equations of the fluid motion. Obviously, the kinematic assumption of 'columnar' motion simplifies the derivation of equations describing the propagation of long waves. Its application to waves propagating in a fluid with a small continuous variation of its depth seems justified; but at the same time, questions arise about the accuracy of the formulation, especially in the case of waves propagating towards sloping beaches. In order to answer this question, we need to compare the results of the theoretical formulation with empirical data. We may expect that the formulation will provide accurate results for small slopes of the fluid bottom. The problem is important from the theoretical and practical points of view, hence the aim of the present paper is to examine the transformation of waves approaching a sloping beach. To this end, experiments conducted in a laboratory flume were an essential part of our investigations. In the experiments, the waves were generated by a piston-type wave maker placed at one end of the flume. A rigid inclined ramp with a slope of 10% was installed at a certain distance from the generator wall. Such an inclination of the ramp was selected for practical reasons associated with the dimensions of the laboratory flume and the inclined plate. The data obtained in the experiments served as a reference for the examination of the approximations in the description of the wave transformation. In the experiments, the parameters of the transforming wave were recorded by a set of wave gauges placed at fixed distances from the wave maker. The run-up of the waves on the sloping ramp was measured using a special conductivity gauge placed on its surface.

2. Theoretical description

Consider a finite fluid domain such as the one shown schematically in Figure 1. The motion of the fluid is induced by the piston-type wave maker starting to move at an initial moment of time. After a finite elapse of time, the generated wave will arrive at the boundary of zero water depth. As the wave propagates, it is transformed as a result of the diminishing water depth. The case shown in Figure 1 corresponds directly to the water wave flume in which the experiments on the transformation of such waves were carried out. In the theoretical description of this phenomenon, we follow the method presented in Szmidt (2006). As in that paper, for the plane

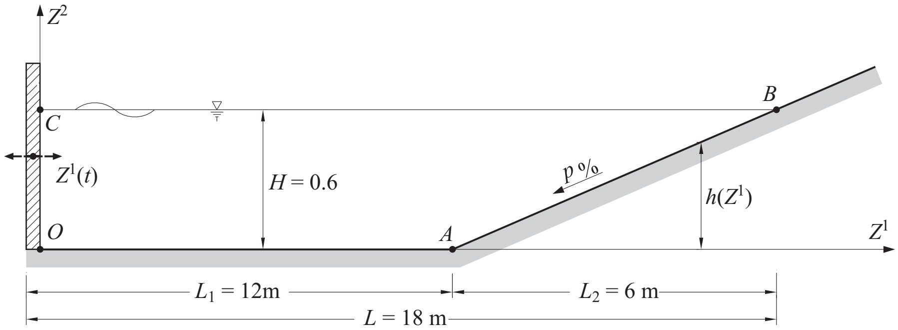

Figure 1. Generation of gravity waves in water of variable depth

problem considered, the material Cartesian coordinates $(Z^1,\ Z^2)$ are chosen as independent variables of the problem. These coordinates designate the fluid particles. The current positions of the fluid particles are described by the spatial Cartesian coordinates $(z^1,\ z^2)$, related to the material ones by

$$z^1(Z^\gamma,\, t) = Z^1 + u(Z^1,\, t), \tag{1}$$
$$z^2(Z^\gamma,\, t) = Z^2 + v(Z^\gamma,\, t), \quad \gamma = 1,\, 2,$$

where $u(Z^1,\ t)$ and $v(Z^1,\ Z^2,\ t)$ are the horizontal and vertical components of the displacement field, and t is time.

Assuming that a vertical material line of fluid particles remains vertical during the entire motion of the fluid, the vertical displacement of an arbitrary particle of the material line is expressed in the form

$$v(Z^1,\, Z^2,\, t) = h(Z^1 + u) - h(Z^1) + \frac{w(Z^1,\, t)}{H - h(Z^1)} \left[Z^2 - h(Z^1) \right], \tag{2}$$

where $h(Z^1)$ defines the bottom elevation in the area of diminishing water depth, and H denotes the constant water depth (see Figure 1).

The first two terms on the right-hand side of the equation describe the rigid displacement of a vertical material line associated with a change in the water depth, and the third term describes linear stretching, corresponding to the vertical displacement $w(Z^1,\ t)$ of the material free surface of the fluid. In this way, the vertical displacement also depends on the horizontal one. The rigid body displacement is approximated by the formula

$$\Delta h = h(Z^1 + u) - h(Z^1) \cong h(Z^1) + h'(Z^1) \times u + \frac{1}{2} h''(Z^1) \times u^2, \tag{3}$$

where the superscripts denote differentiation with respect to Z^1.

For the case of a constant bottom slope ($h' = \mathrm{const}$), the vertical displacement of the water column is a linear function of the horizontal displacement component. With respect to our laboratory experiments, we henceforth confine our attention to the constant bottom slope. Knowing the displacement field, we can calculate the Jacobian of the transformation

$$J = \det[z^i_{,\gamma}] = (1 + u')\left(1 + \frac{w}{H-h}\right), \quad i,\, \gamma = 1,\, 2, \tag{4}$$

where the subscript $\gamma = 1,\, 2$ denotes differentiation with respect to the material coordinates $(Z^1,\, Z^2)$, and hereinafter the prime denotes differentiation with respect to Z^1. For an incompressible fluid the Jacobian is equal to unity, and thus

$$w(Z^1,\, t) = -(H-h)\frac{u'}{1+u'}. \tag{5}$$

From the substitution of equation (5) into relationship (2), we obtain the following equation:

$$v(Z^1,\, Z^2,\, t) = u(Z^1,\, t)h' - \frac{u'}{1+u'}(Z^2 - h). \tag{6}$$

With this formula, the description of the problem has been reduced to a single unknown function $u(Z^1,\, t)$. In order to determine this function, it is necessary to derive equations describing the fluid motion. For the problem discussed here it is convenient to derive a relevant differential equation by means of the Hamilton variational principle. In the variational approach we have to calculate the Lagrangian density function, which is equal to the difference between the kinetic and potential energies of the fluid within the flow domain. Having determined the displacement field, it is a simple task to calculate the potential energy as

$$E_{\mathrm{pot.}} = \rho g \int_0^L \int_{h(Z^1)}^H [z^2(Z^1,\, Z^2,\, t) - Z^2] J dZ^2 dZ^1, \tag{7}$$

where ρ is the fluid density and g is the gravitational acceleration. In a similar way, the kinetic energy is given by

$$E_{\mathrm{kin.}} = \frac{1}{2}\rho \int_0^L \int_{h(Z^1)}^H [(\dot{u})^2 + (\dot{v})^2] J dZ^2 dZ^1, \tag{8}$$

where the dots denote differentiation with respect to time.

Integration of equation (6) with respect to the vertical coordinate yields

$$E_{\text{pot.}} = \frac{1}{2}\rho g H \int\limits_0^L \left[2h'(1-\alpha)u - H(1-\alpha)^2 \frac{u'}{1+u'}\right] dZ^1, \tag{9}$$

where

$$\alpha = \alpha(Z^1) = \frac{h(Z^1)}{H}. \tag{10}$$

Similarly, integration of equation (8) with respect to Z^2 leads to the expression

$$E_{\text{kin.}} = \frac{1}{2}\rho H \int\limits_0^L \left[(1+h'^2)(1-\alpha)(\dot{u})^2 - Hh'(1-\alpha)^2 \frac{\dot{u}\dot{u}'}{(1+u')^2} + \right. \tag{11}$$

$$\left. + \frac{1}{3}H^2(1-\alpha)^3 \frac{(\dot{u}')^2}{(1+u')^4}\right] dZ^1.$$

For the conservative system considered, the variation of the action integral is defined by

$$\delta I = \delta \int\limits_0^{t_a} (E_{\text{kin.}} - E_{\text{pot.}}) dt = 0. \tag{12}$$

Substituting equations (9) and (11) into the latter formula, and then performing simple manipulations, gives

$$\delta I = \frac{1}{2}\rho H \int\limits_0^{t_a} \int\limits_0^L [R_1 \delta \dot{u} + R_2 \delta u' + R_3 \delta \dot{u}'] dZ^1 dt + \tag{13}$$

$$+ \frac{1}{2}\rho g H \int\limits_0^{t_a} \int\limits_0^L [G_1 \delta u' - 2(1-\alpha)h' \delta u] dZ^1 dt = 0,$$

where

$$R_1 = 2(1+h'^2)(1-\alpha)\dot{u} - Hh'(1-\alpha)^2 \frac{\dot{u}'}{(1+u')^2}, \tag{14}$$

$$R_2 = 2Hh'(1-\alpha)^2 \frac{\dot{u}\dot{u}'}{(1+u')^3} - \frac{4}{3}H^2(1-\alpha)^3 \frac{(\dot{u}')^2}{(1+u')^5},$$

$$R_3 = \frac{2}{3}H^2(1-\alpha)^3 \frac{\dot{u}'}{(1+u')^4} - Hh'(1-\alpha)^2 \frac{\dot{u}}{(1+u')^2},$$

$$G_1 = H(1-\alpha)^2 \frac{1}{(1+u')^2}.$$

The terms entering the integrands in equation (13) may be expressed in alternative forms. For example,

$$R_1 \delta\dot{u} = \frac{\partial}{\partial t}(R_1 \delta u) - \dot{R}_1 \delta u. \tag{15}$$

Similar relations can be obtained for the remaining terms. For the fluid motion considered, the arbitrary variation δu vanishes at the end time instants, i.e. for $t = 0$ and $t = t_a$. At the same time, for a prescribed known generator motion, we have $\delta u|_{Z^1=0} = 0$. The condition of stationarity of the action integral leads to the momentum equation

$$\dot{R}_1 + R_2' - \dot{R}_3' + gG_1' + 2gh'(1-\alpha) = 0, \tag{16}$$

and the associated boundary condition is defined by

$$[R_2 - \dot{R}_3 + gG_1]|_{Z^1=L} = 0. \tag{17}$$

One can check that boundary condition (17) is fulfilled at the shore point $Z^1 = L$ identically. In order to express the momentum equation in the form of a differential equation in the unknown function $u(Z^1,\ t)$, relationships (14) have to be substituted into equation (16) and the prescribed differentiation carried out with respect to the space and time coordinates respectively. Simple manipulations give the momentum equation in the following form:

$$\left\{1 + (h')^2\left[1 - \frac{1}{(1+u')^2}\right] - Hh'(1-\alpha)\frac{1}{(1+u')^3}u''\right\}\ddot{u} + \tag{18}$$

$$+\left[H(1-\alpha)h'\frac{1}{(1+u')^4} + \frac{4}{3}H^2(1-\alpha)^2\frac{1}{(1+u')^5}u''\right]\ddot{u}' +$$

$$-\frac{1}{3}H^2(1-\alpha)^2\frac{1}{(1+u')^4}\ddot{u}'' + \frac{4}{3}H^2(1-\alpha)^2\frac{1}{(1+u')^5}\dot{u}''\dot{u}' +$$

$$+\left\{Hh'(1-\alpha)\left[\frac{1}{(1+u')^3}-\frac{2}{(1+u')^5}\right]-\frac{10}{3}H^2(1-\alpha)^2\frac{1}{(1+u')^6}u''\right\}(\dot{u}')^2+$$

$$+g\left\{h'\left[1-\frac{1}{(1+u')^2}\right]-H(1-\alpha)\frac{1}{(1+u')^3}u''\right\}=0.$$

Up to this point no approximations have been introduced. For long water waves and a small bottom slope (h') it seems justified to assume that u' is a small quantity ($u' \ll 1$), and hence, all the fractions with the derivative u' in equation (18) can be approximated by the formula

$$\frac{1}{(1+u')^m} \approx 1 - mu', \quad |u'| \ll 1. \tag{19}$$

By substituting (19) into equation (18) and then neglecting the products of the displacement spatial derivatives and the third and higher order terms entering the equation, we arrive at the final form of the equation of the problem investigated:

$$\ddot{u}[1+2(h')^2u' - Hh'(1-\alpha)u'']+ \tag{20}$$

$$+\ddot{u}'\left[Hh'(1-\alpha)+\frac{4}{3}H^2(1-\alpha)^2u''\right]+$$

$$-\frac{1}{3}H^2\ddot{u}''(1-\alpha)^2+g[2h'u' - H(1-\alpha)u'']=0.$$

The above momentum equation is a non-linear partial differential equation with respect to the two independent variables of the model considered. In order to examine some characteristic features of the equation, it is reasonable to consider its linearized form. Thus, neglecting the second order terms in the equation, one obtains

$$\ddot{u}-\frac{1}{3}H^2(1-\alpha)^2\ddot{u}''+Hh'(1-\alpha)\ddot{u}'+g[2h'u'-H(1-\alpha)u'']=0. \tag{21}$$

Although equation (21) is linear, it has variable coefficients and, therefore, is still difficult to solve analytically. For this reason, in order to find a solution to equation (21), we resort to a discrete formulation by applying the finite difference method. In this discrete approach the derivatives with respect to the material variable Z^1 are replaced by the central finite differences

$$u_k' \cong \frac{1}{2a}(u_{k+1} - u_{k-1}), \tag{22}$$
$$u_k'' \cong \frac{1}{a^2}(u_{k-1} - 2u_k + u_{k+1}),$$

where a is the constant horizontal spacing of the nodal points $k = 0,\ 1,\ 2, ..., N$, with $k = 0$ at the generator wall and $k = N$ at the shoreline $Z^1 = L$ (see Figure 1).

Accordingly, the partial differential equation is transformed into a finite set of ordinary differential equations with respect to the time variable

$$[\mathbf{AM}](\ddot{\mathbf{u}}) + [\mathbf{BM}](\mathbf{u}) = (\mathbf{P}). \tag{23}$$

The matrices $[\mathbf{AM}]$ and $[\mathbf{BM}]$ in the latter equation result from the substitution of definitions (22) into equation (21). The vector $(\mathbf{P})$ in (23) describes a forcing term induced by the generator motion. The integration of the linear system of equations in the time domain will be performed by employing the Wilson θ method (Bathe 1982).

In order to find a solution to the non-linear equation (20), we also apply the finite difference approximation of the spatial derivatives and, instead of the partial differential equation, we will consider a set of the non-linear ordinary differential equations

$$[\mathbf{NAM}]\left(\frac{d^2\mathbf{u}}{dt^2}\right) = (\mathbf{NL}), \tag{24}$$

where the non-linear square 'mass matrix' $[\mathbf{NAM}]$ and the non-linear vector $(\mathbf{NL})$ depend on time and an unknown vector $(\mathbf{u})$.

The non-linear system of differential equations is integrated in the time domain by means of the fourth-order Runge-Kutta method (Björk & Dahlquist 1983). In order to apply the method, the system of equations is expressed in the form

$$\frac{d\mathbf{v}}{dt} = [\mathbf{NAM}]^{-1}(\mathbf{NL}) = \mathbf{FA}(\mathbf{u},\ \mathbf{v},\ t), \tag{25}$$
$$\frac{d\mathbf{u}}{dt} = \mathbf{v}.$$

All components of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{FA}$ in the equations correspond to the set of nodal points $k = 1,\ 2, ...,\ N$ representing the finite fluid domain. Knowing the solution at a given moment of time, say at $t = t^n$, the method allows us to calculate the solution vector at a subsequent instant of time at $t = t^{n+1} = t^n + \Delta t$, where Δt is the time step.

3. Examples of discrete solutions

As mentioned above, the numerical solutions correspond directly to the cases investigated in a laboratory flume. Thus, for the problem shown schematically in Figure 1, we consider the fluid motion induced by a piston-type generator starting to move from rest. In order to avoid discontinuities of field variables, it is assumed that the generator moves in a very smooth manner at the start of its motion; hence, not only the fluid velocity, but also the fluid acceleration are assumed to be both zero at the initial instant of time. Therefore, the motion of the generator wall takes the following form (Wilde & Wilde 2001):

$$u_0(t) = d_u[A(\tau)\cos\omega t + D(\tau)\sin\omega t], \tag{26}$$

where d_u is a dimension unit (in our case one metre), ω is the angular frequency, t is the time measured from the initial instant, τ is a dimensionless time, and

$$A(\tau) = \frac{1}{3!}\tau^3\exp(-\tau), \quad \tau = \eta t, \tag{27}$$
$$D(\tau) = 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right)\exp(-\tau).$$

The parameter η in the above relations is responsible for the growth in time of the generator amplitude. In what follows we will consider the case of $\eta = 2$. With increasing time, harmonic generation with unit amplitude and the adopted frequency is achieved.

Numerical integration in the time domain of equations (25) gives the horizontal displacements of the material nodal points. It should be noted that the momentum equations presented above were derived under the assumption that all the variables of the problem, together with their derivatives, are continuous in space and time. It may happen, however, that a wave climbing the slope will break down. Wave breaking depends on the wavelength, its amplitude and the slope of the inclined plate. In order to estimate the conditions for wave breaking, we use the parameter introduced by Pelinovsky (1991)

$$Br = \frac{\omega^2 R}{g\beta^2}. \tag{28}$$

In this formula, obtained within the framework of shallow water theory, R is the characteristic height of the wave run-up and β is the constant bottom slope. When $Br > 1$, wave breaking is inevitable. On the other hand, the dissipation of energy due to wave breaking slows down shock wave formation, and the critical value of the parameter may be assumed to be $Br \cong 1.5$ for a sea wave breaking (Pelinovsky 1991). With respect to this condition, all the cases considered in this paper correspond to wave breaking. In principle, wave breaking terminates the solution of the problem formulated in a continuum. Fortunately, in the model considered, the problem's solution has been reduced to the horizontal displacement function $u(Z^1,\ t)$, which is also continuous for wave breaking. The last feature enables us to find a solution to the problem, at least within the approximation described by the linear momentum equation (21). Examination of equation (5) shows, however, that in calculating the free surface elevation the most important parameter is the displacement derivative $u'(Z^1,\ t)$. This derivative also enters the non-linear equation (20), derived under the assumption that the derivative is a small quantity. Thus, in order to examine the solutions to equations (21) and (20), we performed test computations for a wave of length $\lambda = 10H$ and a generator amplitude of 0.02 m. Some of the results obtained in the computations are illustrated in the plots in Figures 2 and 3. The plots in Figure 2 show the evolution in time of the horizontal displacements of the chosen material points of the fluid domain. The fluid particles in the area of the smallest water depth undergo the greatest departures from their initial positions. Most of the plots presented in the figure have two different scales of ordinates (one for points $0 \leq Z^1 \leq OA$ and another for points $OA \leq Z^1 \leq OB$). From the plots it may be seen that the fluid displacements in the vicinity of the shore point (B in the figure) are one order of magnitude greater than the displacements of the points corresponding to a greater water depth. The area of large displacements is confined to a small area in the vicinity of the shore point, approximately within 20% of the inclined plate length. The displacements of the remaining fluid particles are of the order of the generator amplitude. Knowing the displacement field, it is a simple task to calculate its derivative with respect to the Z^1-coordinate. The plots of the derivative are shown in Figure 3. As in the case of the

→

Figure 2. Horizontal displacements of fluid particles for a wave of length $\lambda = 10H$ at fixed instants of time. The solid line corresponds to the linear, and the dotted line to the non-linear, momentum equations

Figure 3. Derivative with respect to Z^1 of the horizontal displacements for a wave of length $\lambda = 10H$ at fixed instants of time. The solid line corresponds to the linear, and the dotted line to the non-linear, momentum equations (see page 376)

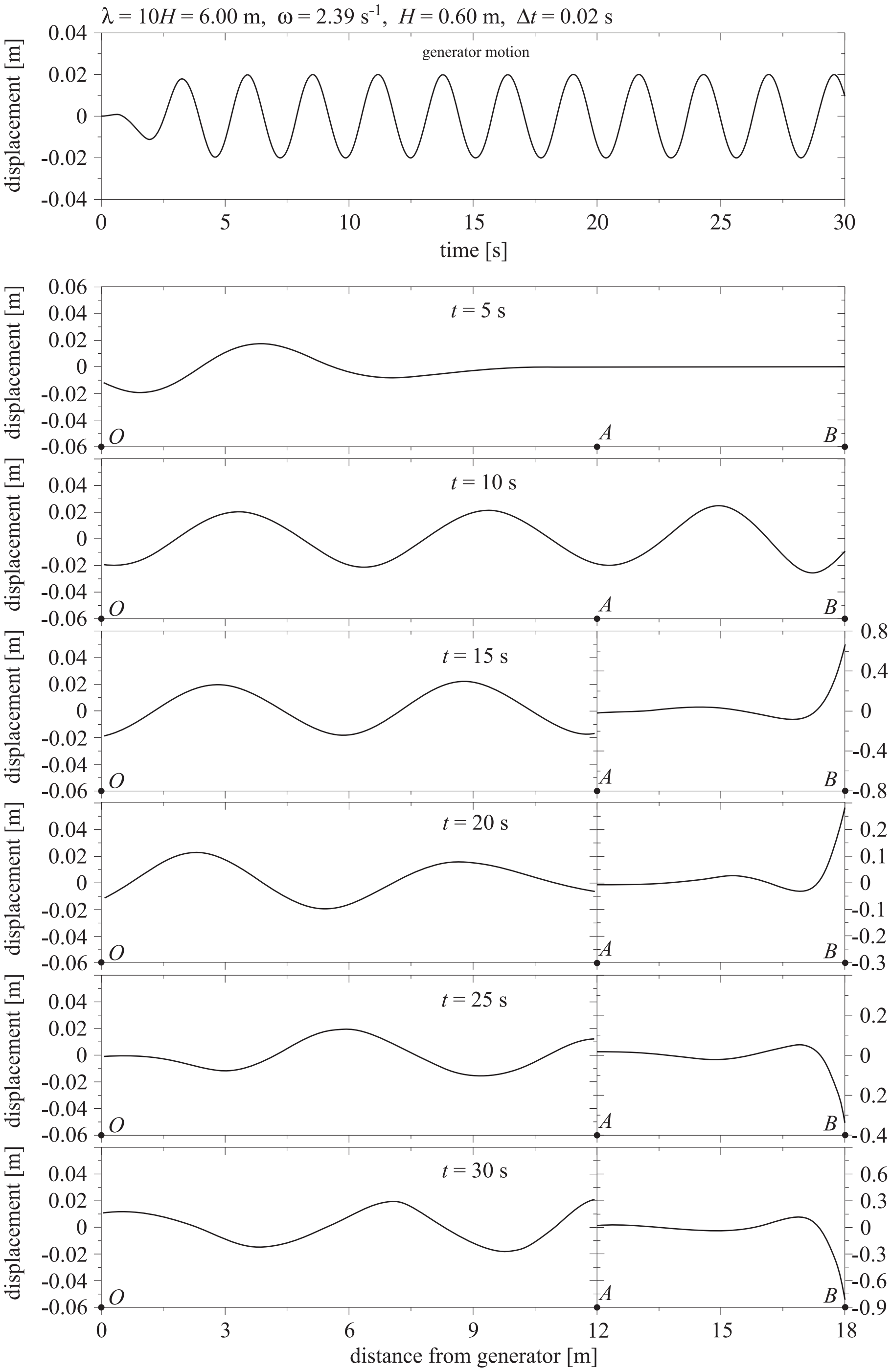
λ = 10H = 6.00 m, ω = 2.39 s-1, H = 0.60 m, Δt = 0.02 s
generator motion
displacement [m]
time [s]
t = 5 s
t = 10 s
t = 15 s
t = 20 s
t = 25 s
t = 30 s
O
A
B
distance from generator [m]

Figure 2.

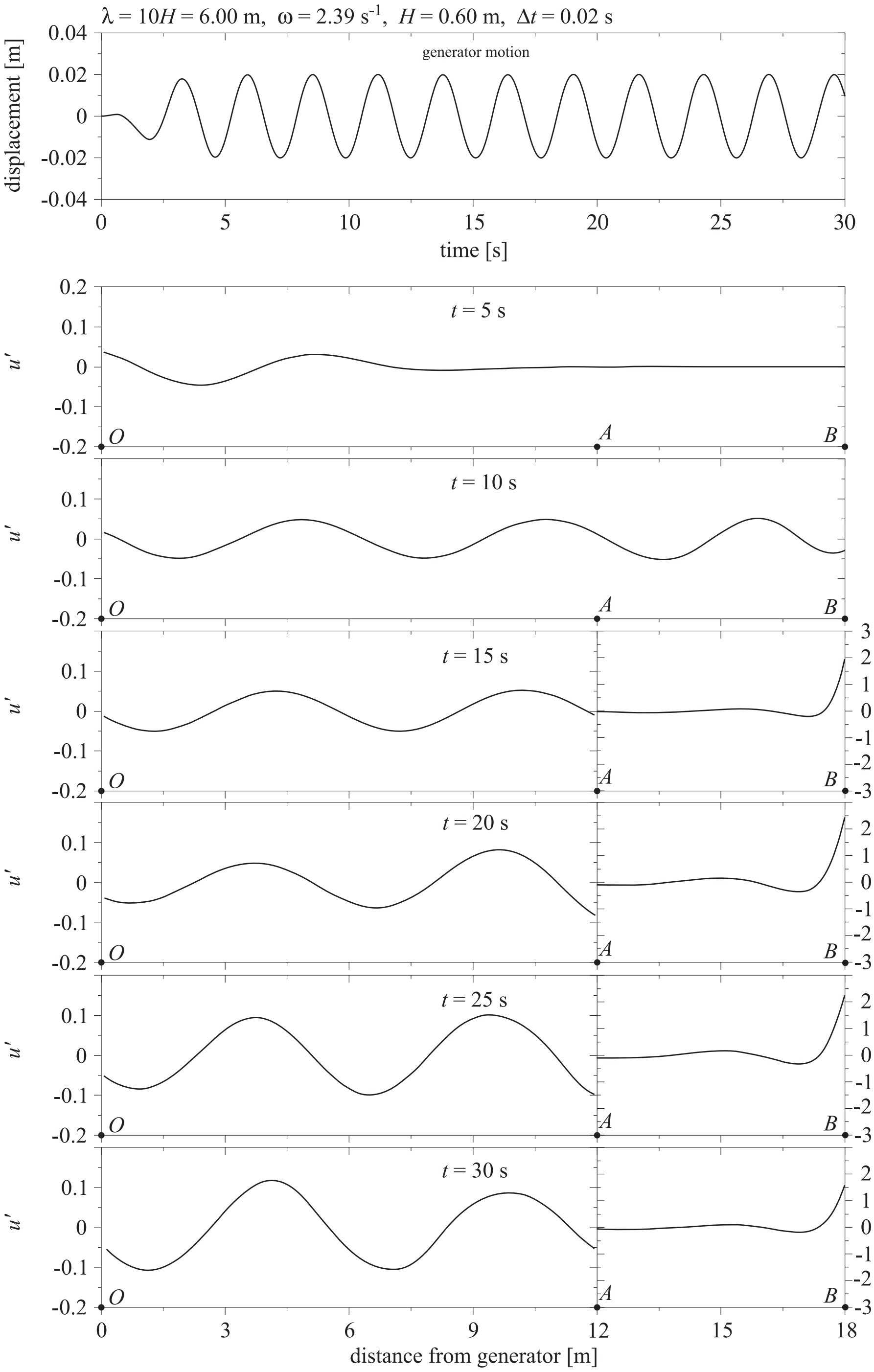
λ = 10H = 6.00 m, ω = 2.39 s-1, H = 0.60 m, Δt = 0.02 s
generator motion
displacement [m]
time [s]
t = 5 s
t = 10 s
t = 15 s
t = 20 s
t = 25 s
t = 30 s
u'
O
A
B
distance from generator [m]

Figure 3.

horizontal displacement, two different scales of ordinates have been used in the derivative plots within the two areas of the fluid domain, i.e. for $0 \leq Z^1 \leq OA$ and $OA \leq Z^1 \leq OB$ respectively. The main features of the displacement distributions and their derivatives are similar. In particular, the derivatives increase as the shore point is approached. With respect to the displacement derivative, questions arise about the accuracy of the numerical results. It may be seen in the plots in Figure 3 that the absolute value of this derivative exceeds unity in the vicinity of the shore point. This means that, in a formal way, the model developed above is capable of calculating the horizontal displacements, but at the same time, it may fail to deliver a proper description of the free surface elevation in the vicinity of the shore point. It may be seen from equation (5) that, for the derivative u' approaching minus one, the elevation (vertical displacement $w(Z^1,\ t)$ of the material free surface) becomes indefinite. In the discrete description of the phenomenon, the indeterminacy of the elevation at a given point may be taken to be the breaking point of a wave climbing the sloping ramp. With smaller ramp slopes, better results of the discrete model can be expected. Numerical tests show that for shorter waves, and for waves of greater heights, the approximate solutions begin to deteriorate, which leads to the breakdown of the computations. Such a case is especially important in the numerical solution of the non-linear momentum equation (24) in which the matrix [**NAM**] may become singular for shorter waves. Therefore, in order to make the further discussion clear, we will confine our attention to long waves of lengths equal to or exceeding $8H$.

4. Experiments in a laboratory flume

The laboratory experiments were conducted in the wave flume at the Institute of Hydro-Engineering of the Polish Academy of Sciences, Gdańsk. The experimental setup is shown schematically in Figure 4. The wave flume is 1.4 m high, 0.6 m wide and 64 m long. The motion of the fluid was induced by a programmable piston-type wave generator placed at one end of the flume. A rigid inclined plate with a slope of 10% was installed at a distance of 12 m from the generator wall. Five wave gauges ($S_i,\ i = 1, ...,\ 5$ in the figure) were mounted alongside the water wave propagation path at fixed distances from the generator plate. The run-up of the waves on the sloping ramp was measured by means of a special conductivity gauge placed on the ramp surface. The gauges registered the water elevation with a sampling frequency of 200 Hz. All the experiments were carried out with a still water depth of 0.6 m. The experiments were conducted for a chosen set of amplitudes and frequencies of the wave generator motion. The set of generator amplitudes was equal to (0.02, 0.04, 0.06) m, and the frequency set

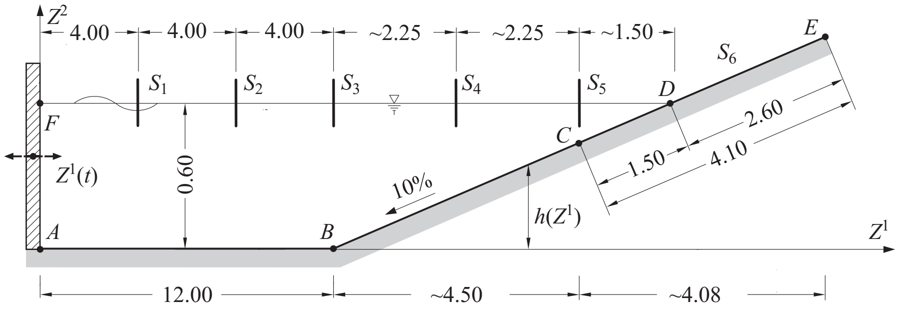

Figure 4. Experimental setup. The surface wave is measured by wave gauges S_i ($i = 1, 2, ..., 5$) and the wave run-up is recorded by conductivity gauge S_6 placed alongside the ramp

corresponded to waves of lengths $\lambda = (4, 6, 8, 10, 12, 14, 16)H$. The motion of the fluid was induced by the generator starting to move at an initial instant of time. With increasing time, the water wave front approached the inclined ramp, after which the wave underwent the transformation associated with diminishing water depth. Finally, the material shore point reached the maximum displacement from its rest position, and then, the reverse motion of the fluid began. The run-down flow assumes the form of a tongue of a water stream of very small thickness. The next wave climbing the sloping ramp meets the earlier wave propagating in the opposite direction, and thus we have a certain interaction between the run-up and the run-down waves on the inclined plate. The interaction causes a discontinuity of the fluid flow. With this emerging discontinuity one can speak of the breaking of waves approaching the shoaling water.

Some of the results obtained in the experiments are illustrated in Figure 5, in which the plots show the distribution in time of the free surface elevations and run-up for waves of length $\lambda = 10H$ and two amplitudes of the generator motion. It may be seen in the plots that the wave elevations measured by the first five gauges at points far from the shore point are of the order of the generation amplitude. The largest displacement is detected by the gauge installed on the ramp. The average in time of the run-up of water waves is greater than zero. In the plot of the run-up it may be seen that the second wave arriving at the slope is considerably larger than the average. The large wave is clearly associated with the transient motion of the generator-fluid system starting to move from rest. The generator

→

Figure 5. Distributions in time of the free surface at points $S_i (i = 1, ...5)$ and the displacement of the shore point S_6 along the slope recorded in experiments for a wave of length $\lambda = 10H$, for two amplitudes of the generator motion

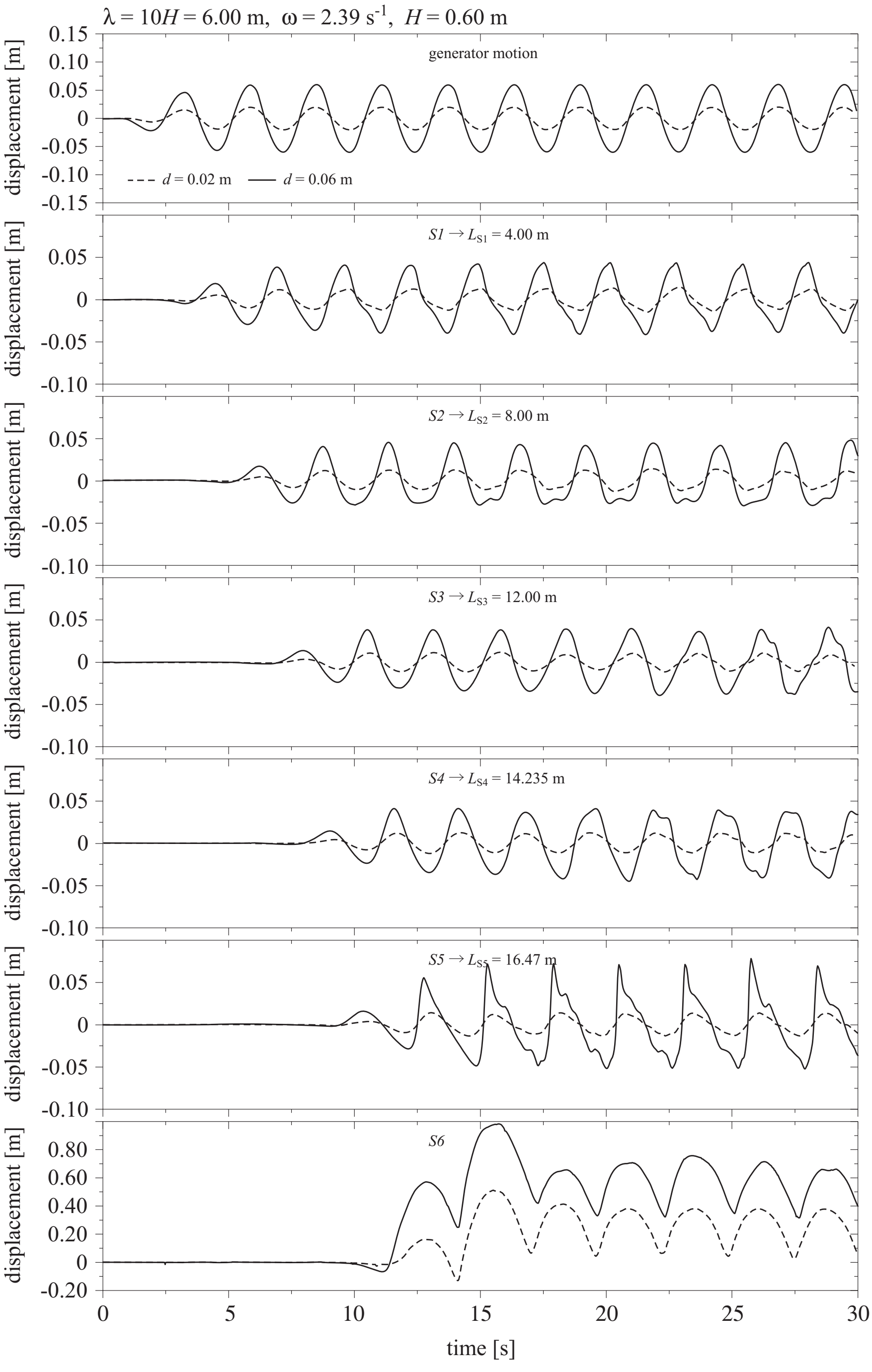
λ = 10H = 6.00 m, ω = 2.39 s-1, H = 0.60 m
generator motion
d = 0.02 m
d = 0.06 m
S1 → LS1 = 4.00 m
S2 → LS2 = 8.00 m
S3 → LS3 = 12.00 m
S4 → LS4 = 14.235 m
S5 → LS5 = 16.47 m
S6
displacement [m]
time [s]

Figure 5.

motion, as described by equation (26) and illustrated in the upper plot in Figure 2, approaches harmonic motion within approximately two periods of wave generation. Moreover, water waves generated by a sinusoidally moving wave maker are always accompanied by a free second order harmonic wave (Madsen 1971, Massel 1982). Such a second harmonic wave (or higher harmonics), however small, may be amplified in the experimental investigations involving resonance to such an extent that it should be taken into account. With increasing generator amplitude, the proportion of higher order components in the description of the surface waves increases.

5. Comparison of theoretical results with experimental data

As mentioned in Sections 2 and 3, the theoretical approach is only a certain approximation of the description of the transformation of water waves propagating over a sloping beach. In particular, the governing equations of the problem have been derived under the assumption that the fluid flow may be regarded as a conservative mechanical system with no dissipation of energy. On the other hand, all the waves considered break on the slope. This means that the transformation of the waves is accompanied by a loss of energy associated not only with frictional forces, but with wave breaking as well (Massel & Pelinovsky 2001). Thus, in order to improve the theoretical description, a dissipation mechanism has to be taken into account. As a description of such a mechanism associated with frictional forces and wave breaking is beyond the scope of this paper, we restrict ourselves to a formal approach to the problem by means of a substitute bottom shear stress added to the momentum equation. Following Dingemans (1997), the bottom shear stress is assumed to take the form of a quadratic law with respect to the velocity, that is,

$$\tau = \varepsilon |\dot{u}| \dot{u}, \tag{29}$$

where ε is a damping coefficient and $\dot{u}$ is the horizontal velocity.

Figure 6. Comparison of the free surface elevation and displacement of the shore point predicted by the computations and those measured in laboratory experiments for waves of length $\lambda = 8H$ (a), $\lambda = 10H$ (b) and $\lambda = 12H$ (c) for a generator amplitude $d = 0.02$ m

Figure 7. Comparison of the free surface elevation and displacement of the shore point predicted by the computations and those measured in laboratory experiments for waves of length $\lambda = 8H$ (a), $\lambda = 10H$ (b) and $\lambda = 12H$ (c) for a generator amplitude $d = 0.06$ m (see page 384)

a

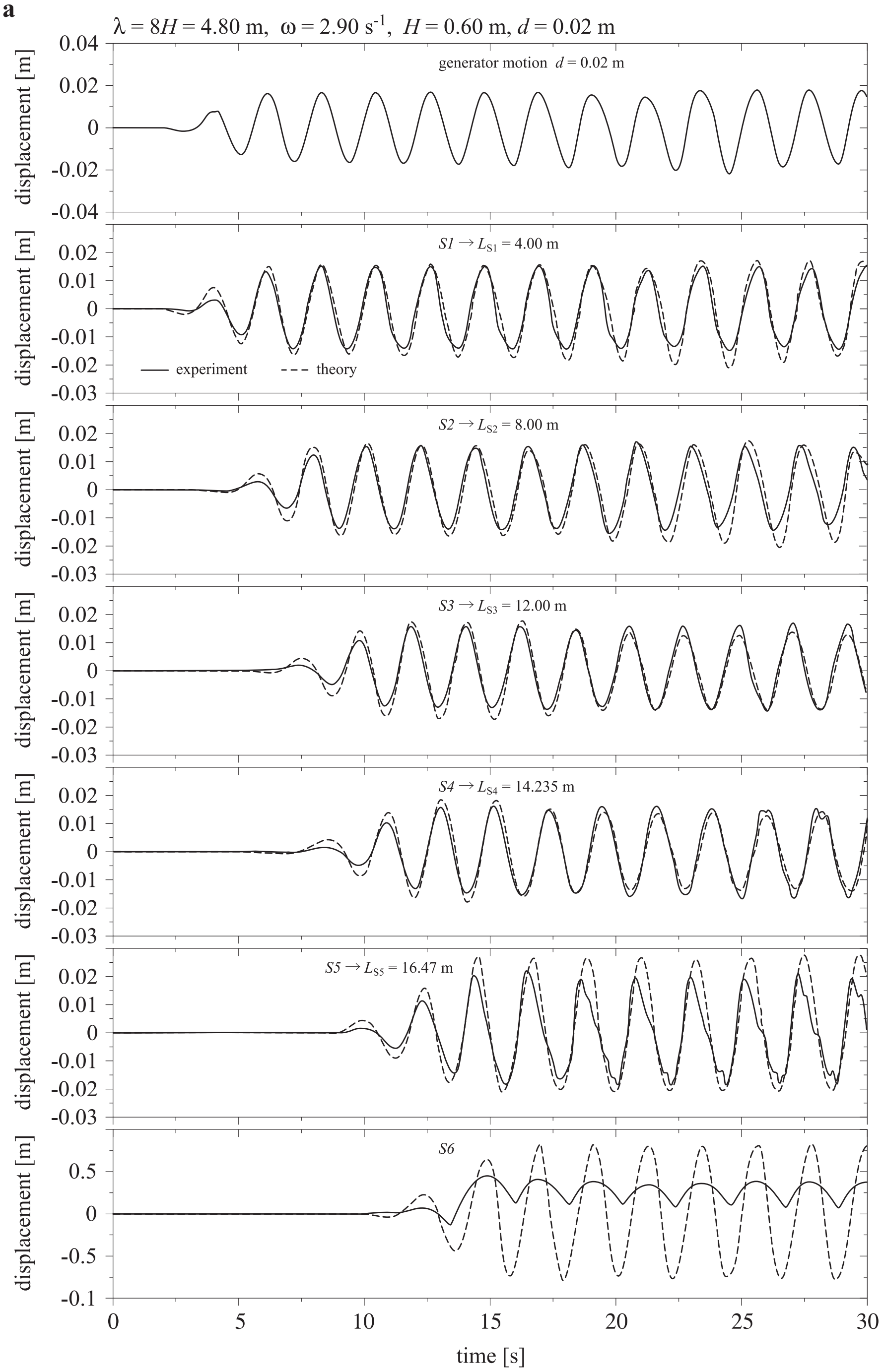
λ = 8H = 4.80 m, ω = 2.90 s-1, H = 0.60 m, d = 0.02 m
generator motion d = 0.02 m
S1 → LS1 = 4.00 m
experiment
theory
S2 → LS2 = 8.00 m
S3 → LS3 = 12.00 m
S4 → LS4 = 14.235 m
S5 → LS5 = 16.47 m
S6
displacement [m]
time [s]

Figure 6.

b

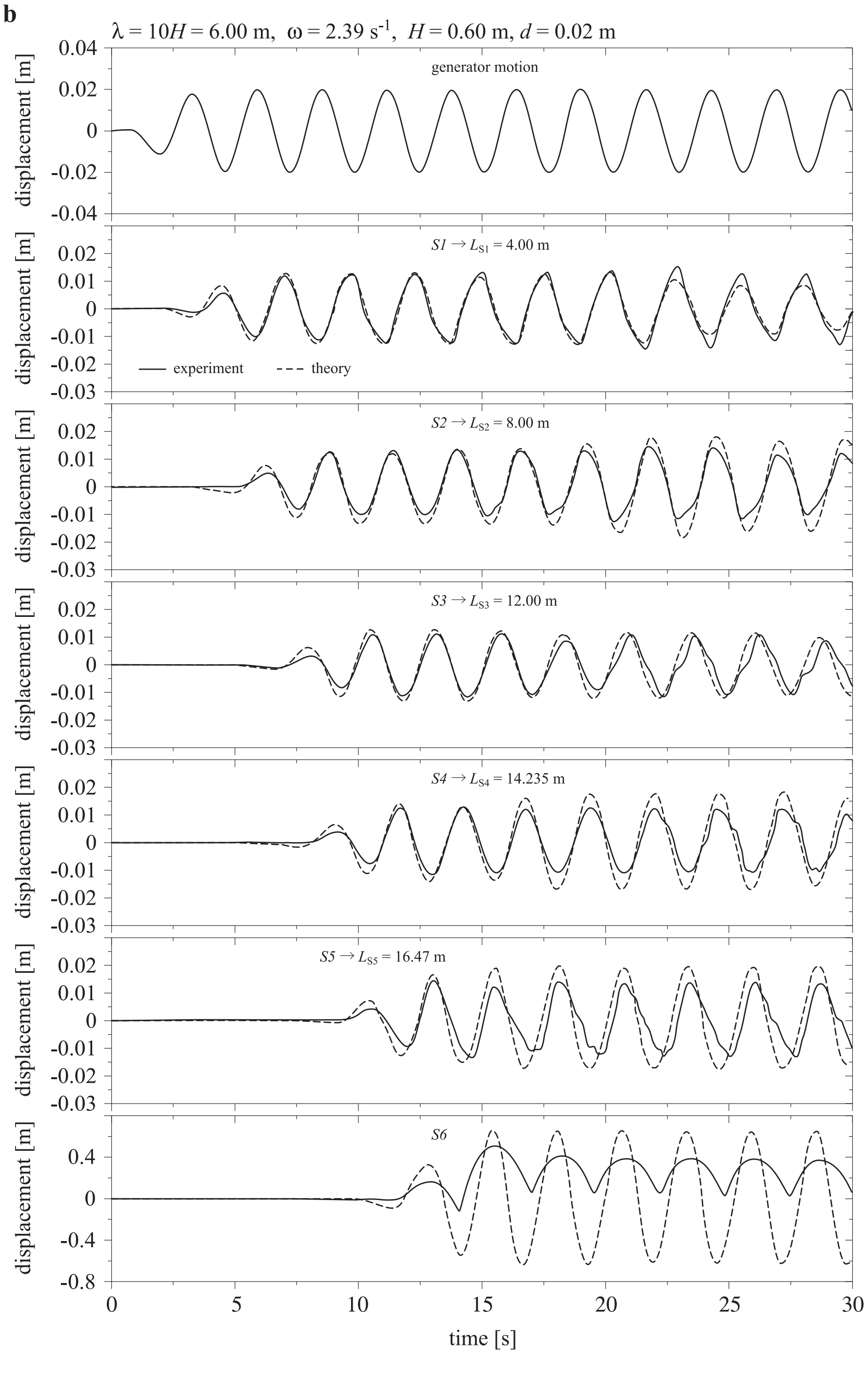

Figure 6. (*continued*)

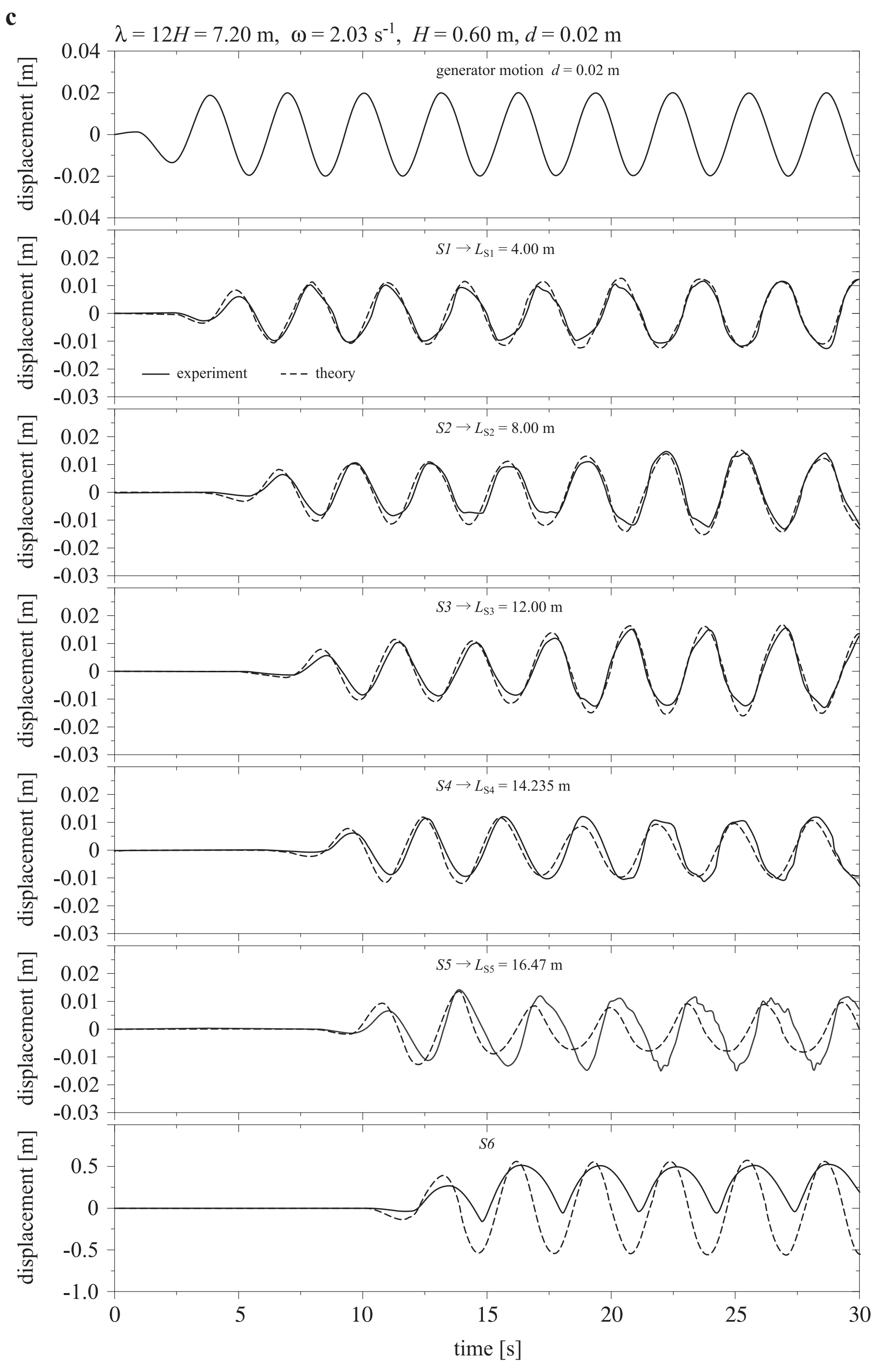

Figure 6. (*continued*)

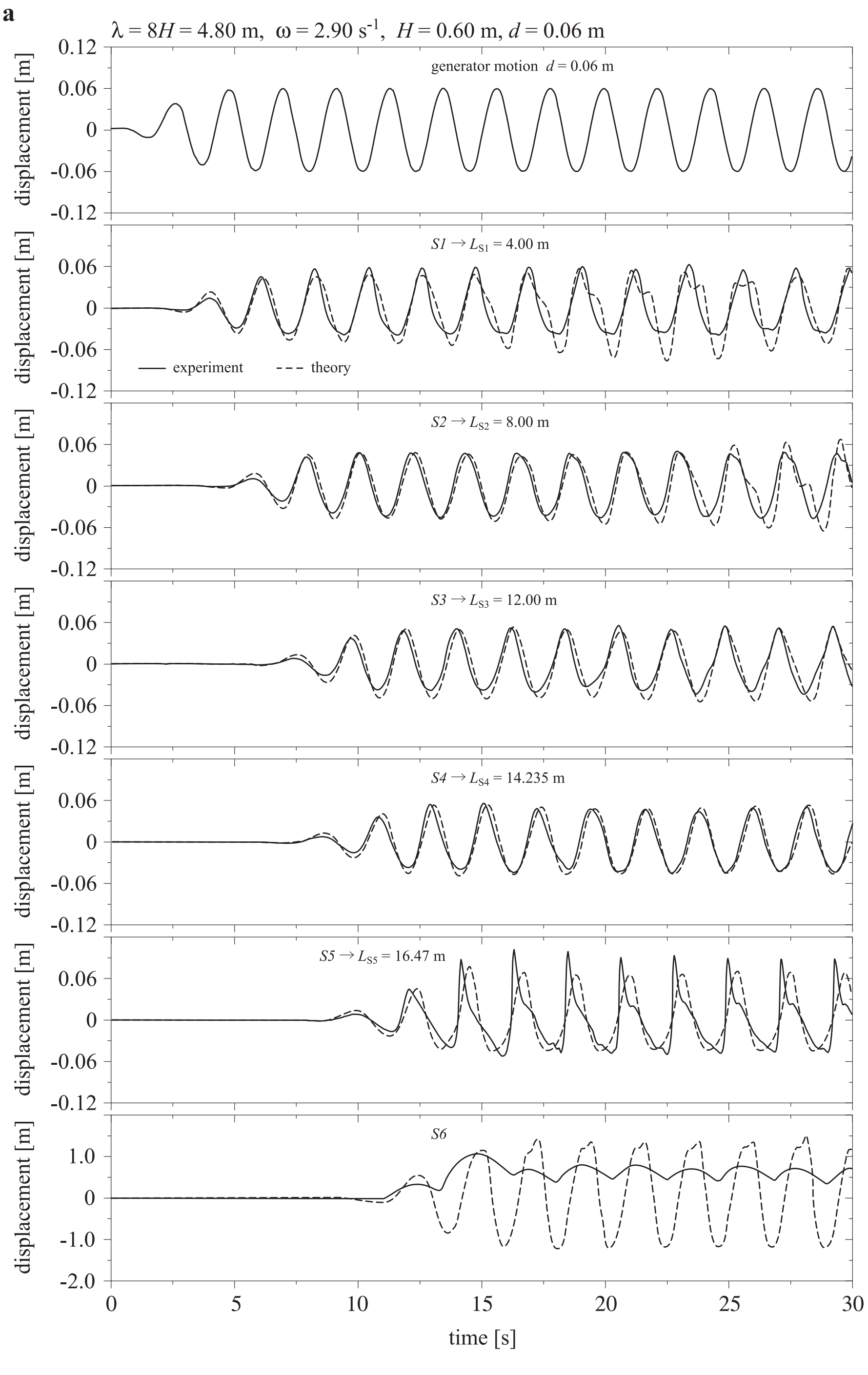
a
λ = 8H = 4.80 m, ω = 2.90 s-1, H = 0.60 m, d = 0.06 m
generator motion d = 0.06 m
S1 → LS1 = 4.00 m
experiment
theory
S2 → LS2 = 8.00 m
S3 → LS3 = 12.00 m
S4 → LS4 = 14.235 m
S5 → LS5 = 16.47 m
S6
displacement [m]
time [s]

Figure 7.

b

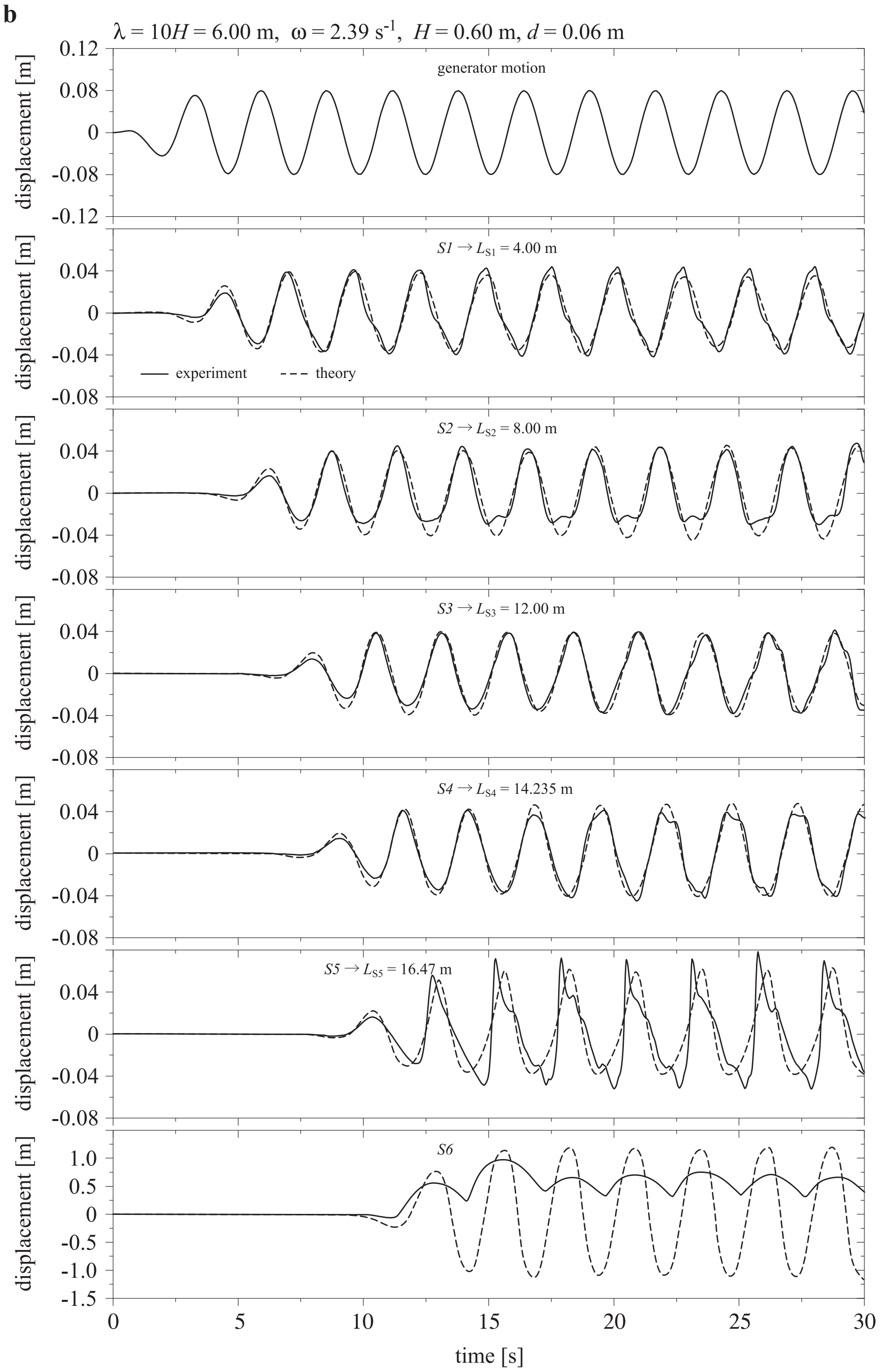

Figure 7. (*continued*)

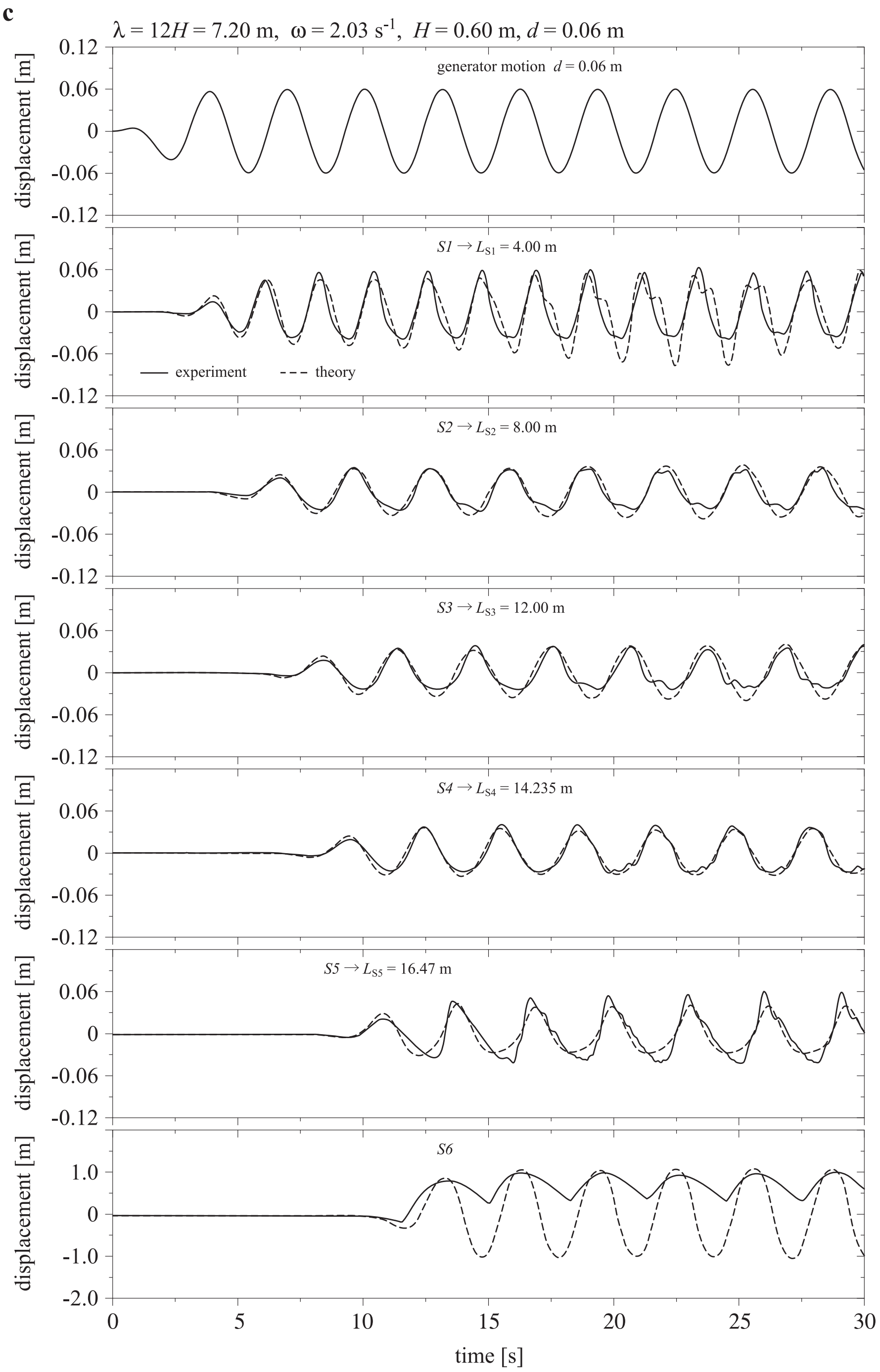

Figure 7. (*continued*)

The above formula is only a crude approximation in the estimation of damping forces associated with long waves propagating in shallow waters. In the model considered here, a dissipation term enters the momentum equation by means of the shear stresses added to the non-linear right hand side of equation (24) (to the vector **NL**). The unknown coefficient in equation (29) depends on the water depth, but in the case under discussion it has been assumed that the coefficient is constant and that shear forces are calculated only at nodal points where the bottom slope is different from zero. In order to assess the effects of damping forces on the theoretical results, numerical tests were performed for selected values of the coefficient. It was found that for values of the coefficient within the range $0.2 \leq \varepsilon \leq 1$ the free surface elevations calculated at the wave gauges S_1–S_5 were almost unchanged. At the same time, there was a $30 - 50\%$ reduction in the wave run-up, dependent on wavelength and height, as compared to solutions for a purely conservative system. Therefore, in our further discussion, we confine our attention to the value $\varepsilon = 1$ in equation (29). Some of the results obtained in computations were compared with data recorded in experiments. The plots in Figures 6 and 7 illustrate the distribution in time of the free surface elevations calculated and measured at the spatial points of the hydraulic flume in which wave gauges were installed. It may be seen in the plots that the theoretical solution, based on the fundamental kinematic assumption of 'columnar' motion, describes satisfactorily the main features of long wave transformation. But at the same time, the theoretical model fails to describe the run-up of the waves with sufficient accuracy. In the area of smaller water depth, higher-order components emerge that cannot be properly described by the model presented above. Such components also emerge in the deeper fluid area for shorter waves, and for waves of higher amplitudes.

6. Concluding remarks

A water wave advancing up a sloping beach undergoes a transformation resulting from the diminishing water depth. The transformation depends on the slope of the beach and the characteristics of the arriving wave. With the water depth diminishing towards the shoreline, the wave grows in steepness, which usually leads to its breaking. In order to estimate the changes in the wave approaching the shore point, both theoretical and experimental investigations were carried out. In the theoretical description we follow the fundamental kinematic assumption of 'columnar motion', which allows us to reduce the two-dimensional problem of long wave propagation to a single unknown variable $u(Z^1,\, t)$ defining the horizontal displacement of a vertical column of water. The associated momentum equation is derived with the aid of Hamilton's principle. The most important parameter in calculating the free surface elevation and the run-up of waves climbing a sloping beach is the derivative of the displacement function with respect to the horizontal coordinate. Comparison of the numerical results with the data obtained in the experiments shows that to obtain a solution of acceptable accuracy the derivative should be a small number ($|u'| < 1$). For the long waves considered in this study, the last condition was satisfied for almost the entire fluid domain, except for a small area in the vicinity of the shore point. Thus, when compared to experiments in a laboratory flume, the theoretical model provides plausible numerical solutions for the free surface elevations at points far away from the shore point but fails to deliver sufficiently good results for the run-up of the waves. In order to calculate the wave run-up a more sophisticated theoretical description is needed. The results presented above indicate that the method based on the 'columnar' assumption of the fluid motion and Hamilton's principle formulated in the material variables has proved to be a useful tool for describing the principal features of long wave transformation. In particular, the theoretical model is also capable of describing fluid motion for waves breaking on the slope. But at the same time, to obtain a better description, the governing equations of motion, derived for a conservative system, should be supplemented by dissipation terms describing the loss of energy associated with friction forces and the breaking of surface waves.

References

Bathe K. J., 1982, *Finite element procedure in engineering analysis*, Prentice-Hall, Upper Saddle River, New Jersey, 1037 pp.

Björk A., Dahlquist G., 1983, *Numerical methods*, PWN, Warszawa, 548 pp., (in Polish).

Carrier G. F., Greenspan H. P., 1958, *Water waves of finite amplitude on a sloping beach*, J. Fluid Mech., 4 (01), 97–109.

Chybicki W., 2006, *Theory of surface waves in water of non-uniform depth*, Wyd. IBW-PAN, Gdańsk, 244 pp., (in Polish).

Dingemans M. W., 1997, *Water wave propagation over uneven bottoms, Part 1 – Linear wave propagation*, World Sci., Singapore–New York, 1061 pp.

Goto C., 1979, *Non-linear equation of long waves in the Lagrangian description*, Coast. Eng. Japan, 22, 1–9.

Kânoğlu U., 2004, *Nonlinear evolution and runup-rundown of long waves over a sloping beach*, J. Fluid Mech., 513, 363–372.

Madsen O. S., 1971, *On the generation of long waves*, J. Geophys. Res., 76 (36), 8672–8683.

Massel S. R., 1982, *On the non-linear theory of paddle generated waves in laboratory channels*, Arch. Hydro-Eng., 39 (3), 183–208, (in Polish).

Massel S. R., Pelinovsky E. N., 2001, *Run-up of dispersive and breaking waves*, Oceanologia, 43 (1), 61–97.

Miles J., Salmon R., 1985, *Weakly dispersive nonlinear gravity waves*, J. Fluid Mech., 157, 519–531.

Pelinovsky E. N., 1991, *Long waves climbing a beach*, Hydrotech. Trans., 54, 81–86.

Shuto N., 1967, *Run-up of long waves on a sloping beach*, Coast. Eng. Japan, 10, 23–37.

Synolakis C. E., 1987, *The run-up of solitary waves*, J. Fluid Mech., 185, 523–545.

Szmidt K., 2006, *Modelling of non-linear long water waves on a sloping beach*, Bull. Polish Acad. Sci., Tech. Sci., 54 (4), 381–389.

Wilde P., Chybicki W., 2004, *Long water waves as a structure – fluid interaction problem*, Arch. Hydro-Eng. Environ. Mech., 51 (2), 95–118.

Wilde P., Wilde M., 2001, *On the generation of water waves in a flume*, Arch. Hydro-Eng. Environ. Mech., 48 (4), 69–83.